Algebraische Topologie 1

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Contents

1 Überblick

- studieren topologische Räume

– Rahmen: Kategorie **Top** der topologischen Räume

- oft Eigenschaften bis auf Homotopieinvarianz: $* \to \mathbb{R}^n$ ist eine Homotopieäquivalenz

– benutzen homotopieinvariante Funktoren:

– $\pi_n(X, *)$ - Homotopiegruppen

– $H_*(X; \mathbb{Z})$ - Homologiegruppen

- Rahmen: Homotopiekategorie **hTop**

Struktur von Argumenten:

– Frage über Objekte in **Top** (Ex: Gibt es einen Homeomorphismus $f : [0, 1] \to S^1$?

– topologische Konstruktion (Ex: Entferne eine Punkt.)

– Homotopieinvariante Frage: (ist $[0, 1] \setminus \{1/2\} \to S^1 \setminus f(\{1/2\})$ eine Homotopieäquivalenz?)

– benutze Homotopieinvariante Funktoren: (Ex: betrachte Mengen der Zusammenhangskomponenten π_0 und zähle)

- typische Fragen die wir (teilweise) benatworten werden (extern:)

1. Sei $f : \mathbb{R}^n \to \mathbb{R}^m$ ein Homöomorphismus. Gilt dann $n = m$?
2. Sei $S^n \to S^m$ eine Homotopieäquivalenz. Gilt dann $n = m$?
3. Sei $f : S^{n-1} \to \mathbb{R}^n$ eine Einbettung. Hat dann $\mathbb{R}^n \setminus f(S^{n-1})$ genau zwei Zusammenhangskomponenten?
4. Gibt es auf S^{n-1} ein Vektorfeld ohne Nullstellen?
5. Ist das Tagentialbpündel von $\mathbb{RP}^n$ trivialisierbar?
6. Für welche n hat $\mathbb{R}^n$ die Struktur einer Divisionsalgebra? Klar: $n = 1, 2, 4$. Welche noch?
7. Jede stetige Abbildung $D^n \to D^n$ hat einen Fixpunkt.
8. Ist jede zu $\mathtt{id}_{S^{n-1}}$ homotope Abbildung surjektiv?

9. Klassifiziere Faserbündel $E \to X$ mit diskreter Faser! (Beschreibung durch Algebra oder Gruppentheorie)
10. Berechne $H^*_{dR}(M)$ für eine glatte Mannigfaltigkeit M.

- interne Frage: Berechnung von Homologie und Homotopie für einfache Räume

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Literatur:

- Hatcher: Algebraic Topology

- verschieden Skripten (Löh, Friedl)

2 Grundlagen

2.1 Topologische Räume, die Kategorie Top

X - Menge

$\mathcal{P}_X$ -Potenzmenge

$\mathcal{T} \subseteq \mathcal{P}_X$

Definition 2.1. *$\mathcal{T}$ ist eine Topologie auf X, wenn folgende Bedingungen erfüllt sind:*

1. *$\mathcal{T}$ ist abgeschlossen under der Bildung von beliebigen Vereinigungen.*
2. *$\mathcal{T}$ ist abgeschlossen unter der Bildung von endlichen Durchschnitten.*
3. $\bigcup_{U \in \mathcal{T}} U = X$

Definition 2.2. *Ein topologischer Raum ist ein Paar $(X, \mathcal{T})$ aus einer Menge X und einer Topologie $\mathcal{T}$ auf X.*

- Elemente von $\mathcal{T}$ heißen offene Mengen

- Komplemente offener Mengen heißen abgeschlossene Mengen

- eine Umgebung A von x in X ist eine Teilmenge, für welche es eine U in $\mathcal{T}$ mit $x \in U \subseteq A$ gibt

Teilräume

X topologischer Raum

- $A \subseteq X$

- $\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}_X\}$ ist induzierte Topologie auf A

– Verifikation: Übungsaufgabe

triviale Beispiele:

- $\mathcal{P}_X$ - diskrete Topologie

– alle Punkte sind offen

- $\{\emptyset, X\}$ chaotische Topologie

– keine nicht-trivialen offenen Mengen

metrische Topologie

- (X, d) - metrischer Raum:

– hat Topologie: $\mathcal{T} := \{U \in \mathcal{P}_X \mid (\forall x \in U \exists \epsilon \in (0, \infty) \mid B(x, \epsilon) \subseteq U)\}$

– Verifikation: Übungsaufgabe

glatte Mannigfaltigkeiten haben unterliegende topologische Räume

$(X, \mathcal{T})$ - topologischer Raum

– Durchschnitt einer Familie abgeschlossener Teilmengen ist abgeschlossen (Komplement ist Vereinigung der offenen Komplemente und damit offen)

- A Teilmenge

– $\bar{A}$ - def. als kleinste abgeschlossen Teilmenge, die A enthält:

Definition 2.3. *Die Teilmenge*

$$\bar{A} := \bigcap_{A \subseteq B, B abgeschl.} B$$

von X heißt der Abschluß von A.

- x in $\bar{A}$, wenn jede Umgebung von x die Menge A nichttrivial schneidet.

$(X, \mathcal{T})$ - topologischer Raum

- die Vereinigung einer Familie offener Teilmengen ist offen

A - Teilmenge

- $\text{int}(A)$ - def. als größte in A enthaltende offene Teilmenge

Definition 2.4. *Die Teilmenge* $\text{int}(A) := \bigcup_{U \subset A, U \in \mathcal{T}} U$ *von X heißt das Innere von A.*

- x in U, wenn es eine in A enthaltende Umgebung von x gibt

Definition 2.5. $\partial A := \bar{A} \setminus \text{int}(A)$ *heißt Rand von A.*

- x in ∂A, wenn jede Umgebung von x sowohl A als auch $X \setminus A$ nicht-trivial schneidet

Beispiel:

$D^n := \{\|x\| \leq 1\}$ in $\mathbb{R}^n$

- $D^n = \bar{D}^n$
- $\mathrm{int}(D^n) = D^n \setminus S^{n-1} = \{\|x\| < 1\}$
- $\partial D^n = S^{n-1} = \{\|x\| = 1\}$

Beispiel:

- 1/3-Kantormenge C in $[0, 1]$
- $C = \bar{C} = \partial C$
- $\mathrm{int}(C) = \emptyset$

Eigenschaften topologischer Räume:

X - topologischer Raum

Definition 2.6. *X ist Hausdorff, falls für je zwei Punkte x, x' mit $x \neq x'$ Elemente U, U' in $\mathcal{T}$ existieren, so daß $x \in U$, $x' \in U'$ und $U \cap U' = \emptyset$ gelten.*

sagen: U und U' trennen x und x'

metrische Räume sind Hausdorff

Teilräume von Hausdorffräumen sind Hausdorff

$(X, \mathcal{T})$ - topologischer Raum

$(U_i)_{i \in I}$ Familie in $\mathcal{T}$

- heißt offene Überdeckung falls $\bigcup_{i \in I} U_i = X$

Definition 2.7. *X is quasi-kompakt, falls für jede offene Überdeckung $(U_i)_{i \in I}$ von X eine endliche Teilmenge I' von I existiert, so daß $\bigcup_{i \in I'} U_i = X$ gilt.*

Definition 2.8. *X heißt kompakt, wenn X Hausdorff und quasi-kompakt ist.*

Beispiele:

beschränkte und abgeschlossen Teilmengen von $\mathbb{R}^n$ sind kompakt

X - topologischer Raum, A Teilmenge

Lemma 2.9. *Wenn A abgeschlossen ist, dann ist A quasi-kompakt.*

Proof.

sei $\mathcal{U} := (U_i)_{i\in I}$ Familie von offenen Teilmenge so daß $A \cap \mathcal{U} := (A \cap U_i)_{i\in I}$ Überdeckung von A ist (A hat die induzierte Topologie)

- $V := X \setminus A$

- $\mathcal{U} \cup (V)$ - Überdeckung von X

- finde darin endliche Teilüberdeckung für X

- finde in $\mathcal{U}$ endliche Teilüberdeckung für A

□

Lemma 2.10. *Wenn X Hausdorff und A (quasi)kompakt ist, dann ist A abgeschlossen.*

Proof. Zu zeigen: $A = \bar{A}$.

- klar $A \subseteq \bar{A}$

- zeigen $\bar{A} \subseteq A$

- x in $\bar{A}$

- Annahme: $x \notin A$:

– wähle für jedes a in A offene Umgebungen U_a von a und V_a von x mit $U_a \cap V_a = \emptyset$ (X Hausdorff)

– $(U_a \cap A)_{a\in A}$ überdeckt A

– wähle endliche Teilmenge B in A mit $\bigcup_{a\in B} U_b \cap A = A$ (A ist kompakt)

– setze $V := \bigcap_{a\in B} V_a$ - offene Umgebung von x

– nach Konstruktion: $V \cap A = \emptyset$ (Widerspruch zu x in $\bar{A}$) □

Morphismen - stetige Abbildungen

- $(X, \mathcal{T})$, $(X', \mathcal{T}')$ - topologische Räume

- $f : X \to X'$ Abbildung der unterliegenden Mengen

Definition 2.11. *f ist stetig, wenn $f^{-1}(\mathcal{T}') \subseteq \mathcal{T}$ gilt.*

Übungsaufgabe: die Komposition von stetigen Abbildungen ist stetig

Definition 2.12. **Top** *ist die Kategorie* **Top** *der topologischen Räume und stetigen Abbildungen.*

erzeugte Topologie:

X - Menge

Beobachtung: der Durchschnitt einer Familie von Topologien auf X ist eine Topologie (Übungsaufgabe)

- $\mathcal{A}$ - Teilmenge von $\mathcal{P}_X$
- $\mathcal{T}(\mathcal{A})$ - def. als kleinste $\mathcal{A}$ enthaltende Topologie

Definition 2.13. *Die von $\mathcal{A}$ erzeugte Topologie ist durch*

$$\mathcal{T}(\mathcal{A}) := \bigcap_{\mathcal{A}\subseteq\mathcal{T}\subseteq\mathcal{P}_X, \mathcal{T}\ Topologie} \mathcal{T}$$

definiert.

durch Abbildungen erzeugte Topologien

X - Menge

- $(Y_i)_{i\in I}$ Familie topologischer Räume
- $(f_i : X \to Y_i)$ - Familie von Abbildungen
- $\mathcal{T}(\bigcup_{i\in I} f^{-1}\mathcal{T}_{Y_i})$ ist kleinste Topologie, so daß f_i für alle i in I stetig ist

- $(g_i : Y_i \to X)_{i\in I}$ - Familie von Abbildungen
- $\{U \in \mathcal{P}_X \mid (\forall i \in I \mid g_i^{-1}(U) \in \mathcal{T}_{Y_i}\}$ ist die größte Topologie auf X so daß g_i für alle i in I stetig ist

Naweise von Stetigkeit auf Erzeugern

- X, Y topologische Räume,
- $f : X \to Y$ Abbildung der unterliegenden Mengen
- $\mathcal{A}$ Teilmenge von $\mathcal{P}_X$
- $\mathcal{T}_Y := \mathcal{T}(\mathcal{A})$

Lemma 2.14. *f ist genau dann stetig, wenn $f^{-1}(\mathcal{A}) \subseteq \mathcal{T}_X$ gilt.*

Proof.

Annahme: f ist stetig

- $f^{-1}(\mathcal{A}) \subseteq f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$

Annahme: $f^{-1}(\mathcal{A}) \subseteq \mathcal{T}_X$

- betrachte auf Y größte Topologie $\mathcal{T}$ so daß $(X, \mathcal{T}(f^{-1}(\mathcal{A})) \to (Y, \mathcal{T})$ stetig ist
- nach Konstruktion $\mathcal{A} \subseteq \mathcal{T}$, also auch $\mathcal{T}_Y = \mathcal{T}(\mathcal{A}) \subseteq \mathcal{T}$
- aus Annahme folgt: $\mathcal{T}(f^{-1}(\mathcal{A})) \subseteq \mathcal{T}_X$
- also $f^{-1}(\mathcal{T}_Y) \subseteq f^{-1}(\mathcal{T}) \subseteq \mathcal{T}(f^{-1}(\mathcal{A})) \subseteq \mathcal{T}_X$ (mittlere Inklusion nach Konstruktion von $\mathcal{T}$)
- das ist die Stetigkeit von $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ □

$f : X \to Y$ Morphismus in **Top**

Lemma 2.15. *Wenn X quasi-kompakt ist, dann ist $f(X)$ quasi-kompakt.*

Proof. $(U_i)_{i \in I}$ - offene Überdeckung von $f(X)$

- $(f^{-1}(U_i))_{i \in I}$ ist offene Überdeckung von X
- finde endliche Teilmenge I' von I mit $\bigcup_{i \in I'} U_i = X$
- dann $\bigcup_{i \in I'} U_i = f(X)$ □

Lemma 2.16. *Wenn f bijektiv, X quasi-kompakt und Y Hausdorff ist, dann ist f ein Isomorphismus.*

Proof. f stetig: $f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$

zu zeigen: f ist offen: $f(\mathcal{T}_X) \subseteq \mathcal{T}_Y$

- f ist bijektiv under erhält damit Komplemente
- gzz: wenn A abgeschlossen in X ist, dann ist $f(A)$ abgeschlossen in Y

– A abgeschlossen $\Rightarrow$ A ist quasi-kompakt $\Rightarrow$ $f(A)$ ist quasi-kompakt $\Rightarrow$ $f(A)$ ist abeschlossen

□

2.2 Vollständigkeit und Kovollständigkeit

Einschub: Erinnerungen adjungierte Funktoren

- $\mathbf{C}, \mathbf{D}$ - Kategorien

- $L : \mathbf{C} \to \mathbf{D}$ und $R : \mathbf{D} \to \mathbf{C}$ Funktoren

Definition 2.17. *Eine Adjunktion (L, ϕ, R) ist eine bi-natürliche Bijektion*

$$\phi_{C,D} : \mathtt{Hom}_{\mathbf{D}}(L(C), D) \xrightarrow{\cong} \mathtt{Hom}_{\mathbf{C}}(C, R(D)) \ .$$

- bi-natürlich heißt:

– für jeden Morphismus $f : C \to C'$ in $\mathbf{C}$ und Objekt D in $\mathbf{D}$ kommutiert

$$\begin{array}{ccc} \mathtt{Hom}_{\mathbf{D}}(L(C'), D) & \xrightarrow{\phi_{C',D}} & \mathtt{Hom}_{\mathbf{C}}(C', R(D)) \\ \downarrow{\scriptstyle L(f)^*} & & \downarrow{\scriptstyle f^*} \\ \mathtt{Hom}_{\mathbf{D}}(L(C), D) & \xrightarrow{\phi_{C,D}} & \mathtt{Hom}_{\mathbf{C}}(C, R(D)) \end{array}$$

und für jeden Morphismus $g : D \to D'$ in $\mathbf{D}$ und Objekt C in $\mathbf{C}$ kommutiert

$$\begin{array}{ccc} \mathtt{Hom}_{\mathbf{D}}(L(C), D) & \xrightarrow{\phi_{C,D}} & \mathtt{Hom}_{\mathbf{C}}(C, R(D)) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle R(g)_*} \\ \mathtt{Hom}_{\mathbf{D}}(L(C), D') & \xrightarrow{\phi_{C,D'}} & \mathtt{Hom}_{\mathbf{C}}(C, R(D')) \end{array}$$

- schreiben $L : \mathbf{C} \leftrightarrows \mathbf{D} : R$ oder $L : \mathbf{C} \overset{\phi}{\leftrightarrows} \mathbf{D} : R$ oder (L, ϕ, R) oder

Eindeutigkeit von Adjungierten

- sei L gegeben

– wenn eine Adjunktion (L, ϕ, R) existiert, dann ist das Paar (ϕ, R) eindeutig bis auf eindeutige Isomorphie (Übungsaufgabe: Überlegen, was das genau bedeutet!):

– in der Tat, für D in $\mathbf{D}$ stellt $R(D)$ den Funktor

$$\mathbf{C}^{\mathrm{op}} \ni C \mapsto \mathtt{Hom}_{\mathbf{D}}(L(C), D) \in \mathbf{Set}$$

dar

– Aussage folgt mit dem Yoneda Lemma

Details: Übungsaufgabe

- sei R gegeben:

– wenn eine Adjunktion (L, ϕ, R) existiert, dann ist (L, ϕ) eindeutig bis auf eindeutige Isomorphie:

Examples of adjunctions from linear algebra:

consider K - a field

$$K[-] : \mathbf{Set} \leftrightarrows \mathbf{Vect}_K : \text{underlying set} .$$

$K[X]$ - K-vector space generated by X (with basis X)

- bi-natural bijection

$$\phi : \mathtt{Hom}_{\mathbf{Vect}_K}(K[X], V) \stackrel{\cong}{\to} \mathtt{Hom}_{\mathbf{Set}}(X, V)$$

– $\to$ - restriction to X

- $\leftarrow$ - linear extension

for V in $\mathbf{Vect}_K$:

$$- \otimes_K V : \mathbf{Vect}_K \leftrightarrows \mathbf{Vect}_K : \mathtt{Hom}_K(V, -)$$

- bi-natural bijection

$$\phi : \mathtt{Hom}_{\mathbf{Vect}_K}(W \otimes V, Z) \stackrel{\cong}{\to} \mathtt{Hom}_{\mathbf{Vect}_K}(W, \mathtt{Hom}_K(V, Z))$$

– $\to$ - $f \mapsto (w \mapsto (v \mapsto f(w \otimes v)))$

–$\leftarrow$ - $g \mapsto ((w \otimes v) \mapsto g(w)(v))$

Ende des Einschubs über Adjunktionen

haben "vergiß Topologie" Funktor

$$\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$$

- haben Inklusion

$$\mathtt{Hom}_{\mathbf{Top}}(X, Y) \subseteq \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(X), \mathcal{F}(Y))$$

Lemma 2.18. *Der Funktor $\mathcal{F}$ hat einen linksadjungierten und einen rechtsadjungierten.*

Proof. - linksadjungierter Funktor

$$(-)_{\mathrm{disc}} : \mathbf{Set} \to X \ , \quad X \mapsto X_{\mathrm{disc}}(X, \mathcal{P}_X)$$

$$(-)_{disc} : \mathbf{Set} \leftrightarrows \mathbf{Top} : \mathcal{F}$$

- haben offensichtliche binatürliche Bijektion (Gleichheit)

$$\mathtt{Hom}_{\mathbf{Top}}(X_{\mathrm{disc}}, Y) \cong \mathtt{Hom}_{\mathbf{Set}}(X, \mathcal{F}(Y))$$

für Y in **Top** und X in **Set**

- rechtsadjungierter Funktor

$$(-)_{\mathrm{chaot}} : \mathbf{Set} \to \mathbf{Top} \ , \quad X \mapsto (X, \{\emptyset, X\})$$

$$\mathcal{F} : \mathbf{Top} \leftrightarrows \mathbf{Set} : (-)_{\mathrm{chaot}}$$

- haben offensichtliche binatürliche Bijektion (Gleichheit)

$$\mathtt{Hom}_{\mathbf{Top}}(Y, X_{\mathrm{chaot}}) \cong \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(Y), X)$$

für Y in **Top** und X in **Set** □

Einschub: Limits and Colimits

consider **C**,**I** categories

- assame that **I** is small, i.e. Ob(**I**) is a set

Definition 2.19. *The objects of* **Fun**(**I**, **C**) *are called diagrams of shape* **I**.

Example:

for C in **C** we have a constant diagram $\underline{C}$ with value C

- $C : \mathbf{I} \to \mathbf{C}$ a diagram
- C' - an object of **C**

- $\mathbf{I} := \bullet \leftarrow \bullet \to \bullet$

– **I**- diagrams are of the form

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \\ A & & \end{array}$$

- $\mathbf{I} := \bullet \to \bullet \leftarrow \bullet$

– **I**- diagrams are of the form

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & C \end{array}$$

consider a diagram $C : \mathbf{I} \to \mathbf{C}$

C^t in $\mathbf{C}$

Definition 2.20. *A cone over C with tip C^t is a morphism $\underline{C^t} \to C$ in $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$.*

write this as

$$C^t \triangleleft C$$

- explicitly:

- $C^t \triangleleft C$ is given by a collection of morphisms $(e_i : C^t \to C(i))_{i \in \mathbf{I}}$ in $\mathbf{C}$ such that

$$\begin{array}{ccccc} & & C^t & & \\ & \swarrow^{e_i} & & \searrow^{e_{i'}} & \\ C(i) & & \xrightarrow{C(\phi)} & & C(i') \end{array}$$

commutes for every morphism $\phi : i \to i'$ in $\mathbf{I}$

consider $F : \mathbf{C} \to \mathbf{D}$ - a functor

- $C : \mathbf{I} \to \mathbf{C}$ a diagram

– get $F(C) := F \circ C : \mathbf{I} \to \mathbf{D}$

- start with cone $C^t \triangleleft C$ over C

- get cone $F(C^t) \triangleleft F(C)$ over $F(C)$

Example:

T object in $\mathbf{C}$

- $\mathtt{Hom}_{\mathbf{C}}(T, -) : \mathbf{C} \to \mathbf{Set}$

- get cone $\mathtt{Hom}_{\mathbf{C}}(T, C^t) \triangleleft \mathtt{Hom}_{\mathbf{C}}(T, C)$

consider two cones $C^t \triangleleft C$ and $C^{t\prime} \triangleleft C$ over C

Definition 2.21. *A morphism of cones $f : C^t \triangleleft C \to C^{t\prime} \triangleleft C$ over C is a morphism $f : C^t \to C^{t\prime}$ in* $\mathbf{C}$ *such that*

$$\begin{array}{ccc} C^t & \xrightarrow{\quad f \quad} & C^{t\prime} \\ & {\scriptstyle e_i} \searrow \quad \swarrow {\scriptstyle e'_i} & \\ & C(i) & \end{array}$$

commutes for every i in $\mathbf{I}$.

get category of cones over C

Lemma 2.22. *A limit cone is a final object in the category of cones over C. Its tip is called a limit of C and denoted by $\mathtt{lim}_{\mathbf{I}}\, C$.*

- a limit cone is unique up to unique isomorphism

Definition 2.23. *A square*

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is called a pull-back diagram (cartesian square) if it is a limit cone over

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & C \end{array}$$

- note that we omit the to write the diagonal, which is redundant information

explicit limit cones in **Set**

$X : \mathbf{I} \to \mathbf{Set}$

- define set

$$\mathop{\mathtt{lim}}_{\mathbf{I}} X := \{(x_i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} X(i) \mid (\forall \phi : i \to i' \in \mathrm{Mor}(\mathbf{I}) \mid X(\phi)(x_i) = x_{i'})\} \subseteq \prod_{i\in\mathbf{I}} X(i)$$

- define family $(e_j : \mathtt{lim}_{\mathbf{I}}\, X \to X(j))_{j\in\mathbf{I}}$ of maps

$$e_j : \mathop{\mathtt{lim}}_{\mathbf{I}} X \to X(j)\ , \quad (x_i)_{i\in\mathbf{I}} \to x_j$$

Lemma 2.24. $\mathtt{lim}_{\mathbf{I}} X \lhd X$ *is a limit cone.*

Proof. Übungsaufgabe $\square$

- we give now explicit description of limits

$C : \mathbf{I} \to \mathbf{C}$ - diagram

Lemma 2.25. *A limit cone over C is a cone $\mathtt{lim}_{\mathbf{I}} C \lhd C$ (given by $(e_i)_{i\in\mathbf{I}}$) such that for every T in $\mathbf{C}$ the induced map*

$$\mathtt{Hom}_{\mathbf{C}}(T, \mathop{\mathtt{lim}}_{\mathbf{I}} C) \to \mathop{\mathtt{lim}}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{C}}(T, C) \ , \quad f \mapsto (e_i \circ f)_{i\in\mathbf{I}}$$

is a bijection.

note that right-hand side uses explicit description of limits in **Set** as a subset of the product

Proof. Übungsaufgabe $\square$

limit as a right-adjoint functor

- **I** - small category
- **C** category

Lemma 2.26. *If* **C** *admits all limits of shape* **I**, *then we have an adjunction*

$$\underline{(-)} : \mathbf{C} \leftrightarrows \mathbf{Fun}(\mathbf{I}, \mathbf{C}) : \mathop{\mathtt{lim}}_{\mathbf{I}} \ .$$

Proof.

consider C in $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$

- assume $\mathtt{lim}_{\mathbf{I}} C \rhd C$ is limit cone
  – let $(e_i)_{i\in\mathbf{I}}$ be the family of structure maps
- for C' in $\mathbf{C}$:

$$\mathtt{Hom}_{\mathbf{C}}(C', \mathop{\mathtt{lim}}_{\mathbf{I}} C) \to \mathtt{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(\underline{C'}, C) \ , \quad f \mapsto (e_i \circ f)_{i\in\mathbf{I}}$$

is bijection and natural in C'

– use here $\mathtt{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(\underline{C'}, C) = \mathtt{lim}_{\mathbf{I}} \mathtt{Hom}(C', C)$ as subsets of $\prod_{i\in\mathbf{I}} \mathtt{Hom}_{\mathbf{C}}(C', C(i))$

- hence the functor

$$\mathbf{C}^{\mathrm{op}} \to \mathbf{Set} \ , \quad C' \mapsto \mathtt{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(\underline{C'}, C)$$

is representable by $\texttt{lim}_{\mathbf{I}}\, C$

- this implies the existence of the right adjoint of $\underline{(-)}$ with the claimed values

$\square$

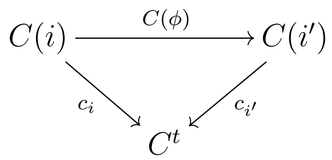
$C : \mathbf{I} \to \mathbf{C}$

C^t in $\mathbf{C}$

Definition 2.27. *A cone under C with tip C' is a morphism $C \to \underline{C^t}$ in $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$.*

- write $C \triangleright C^t$
- explicitly:

– $C \triangleright C^t$ is given by family of morphisms $(c_i : C(i) \to C^t)_{i \in \mathbf{I}}$ in $\mathbf{C}$ such that



commutes for every morphism $\phi : i \to i$ in $\mathbf{I}$

- $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ - functor
- $C \triangleright C^t$ - cone under C
- get cone $F(C^t) \triangleleft F(C)$ over $F(C)$

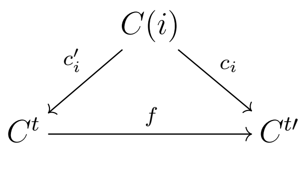
Example:

T in $\mathbf{C}$

- $\texttt{Hom}_{\mathbf{C}}(-, T) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$
- get $\texttt{Hom}_{\mathbf{C}}(C^t, T) \triangleleft \texttt{Hom}_{\mathbf{C}}(C, T)$ over $\texttt{Hom}_{\mathbf{C}}(C, T)$

consider two cones $C \triangleright C^t$ and $C \triangleright C^{t\prime}$ under C

Definition 2.28. *A morphism of cones $f : C \triangleright C^t \to C \triangleright C^{t\prime}$ under C is a morphism $f : C^t \to C^{t\prime}$ in $\mathbf{C}$ such that*



commutes for every i in $\mathbf{I}$.

get category of cones under C

Lemma 2.29. *A colimit cone is an initial object in the category of cones under C. Its tip is called a colimit of C and denoted by $\mathtt{colim}_{\mathbf{I}} C$.*

- a colimit cone is unique up to unique isomorphism

Definition 2.30. *A square*

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \sqcup_C B \end{array}$$

is a push-out diagram (cocartesian square) if it is a colimit cone under

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \\ A & & \end{array}$$

- omit to write the diagonal (redundant information)

explicit description

Lemma 2.31. *A colimit cone under C is a cone $(C \triangleright \mathtt{colim}_{\mathbf{I}} C)$ (given by $(c_i)_{i\in\mathbf{I}}$) such that for every T in $\mathbf{C}$ the induced map*

$$\mathtt{Hom}_{\mathbf{C}}(\mathop{\mathtt{colim}}_{\mathbf{I}} C, T) \to \mathop{\mathtt{lim}}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{C}}(C, T) \ , \quad g \mapsto (g \circ c_i)_{i\in\mathbf{I}}$$

is a bijection.

Proof. Übungsaufgabe ☐

I small category

C - category

Lemma 2.32. *If $\mathbf{C}$ admits all colimits of shape $\mathbf{I}$, then we have an adjunction*

$$\mathop{\mathtt{colim}}_{\mathbf{I}} : \mathbf{Fun}(\mathbf{I}, \mathbf{C}) \leftrightarrows \mathbf{C} : \underline{(-)} \ .$$

Proof. apply Lemma 2.26 to $\mathbf{C}^{\mathrm{op}}$ ☐

Example: colimits in **Set**

$X : \mathbf{I} \to \mathbf{Set}$ diagram

- define the set

$$\operatorname*{\mathtt{colim}}_{\mathbf{I}} X := \bigsqcup_{i \in \mathbf{I}} X(i)/\sim$$

– where $\sim$ is generated by $(x, \phi(x))$ für alle Morphismen $\phi : i \to i'$ in $\mathbf{I}$ und x in $X(i)$

- define family of morphisms $(c_i : X(i) \to \mathtt{colim}_{\mathbf{I}} X)_{i \in \mathbf{I}}$

$$c_i : X(i) \to \operatorname*{\mathtt{colim}}_{\mathbf{I}} X \ , \quad x \mapsto [x]$$

Lemma 2.33. *$X \triangleright \mathtt{colim}_{\mathbf{I}} X$ is a colimit cone.*

Proof. Übungsaufgabe □

$F : \mathbf{C} \to \mathbf{D}$ a functor

$C : \mathbf{I} \to \mathbf{C}$ - a diagram

- assume that limit cones below exist

- get unique morphism of cones (since limit cones are final)

$$(F(\operatorname*{\mathtt{lim}}_{\mathbf{I}} C) \triangleleft F(C)) \to (\operatorname*{\mathtt{lim}}_{\mathbf{I}} F(C) \triangleleft F(C))$$

Definition 2.34. *F preserves limits if the canonical morphism $F(\mathtt{lim}_{\mathbf{I}} C) \to \mathtt{lim}_{\mathbf{I}} F(\mathbf{C})$ is an equivalence for every diagram C in $\mathbf{C}$.*

- one can restrict the shapes of the diagrams and state that F preserves limits of a given class of shapes

- assume that colimit cones below exist

- get unique morphism of cones (since colimit cones are initial)

$$(F(C) \triangleright \operatorname*{\mathtt{colim}}_{\mathbf{I}} F(C)) \to (F(C) \triangleright F(\operatorname*{\mathtt{colim}}_{\mathbf{I}} C))$$

Definition 2.35. *F preserves colimits if the canonical morphism $\mathtt{colim}_{\mathbf{I}} F(\mathbf{C}) \to F(\mathtt{colim}_{\mathbf{I}} C)$ is an equivalence for every diagram C in $\mathbf{C}$.*

- one can restrict the shapes of the diagrams and state that F preserves colimits of a given class of shapes

Lemma 2.36.

1. *Right adjoints preserve limit cones.*
2. *Left adjoints preserve colimit cones.*

Proof. $D : \mathbf{I} \to \mathbf{D}$ - diagram

- $\mathtt{lim}_{\mathbf{I}} D \lhd D$ - limit cone

- $R : \mathbf{D} \to \mathbf{C}$ functor with left-adjoint L

- T arbitrary in $\mathbf{C}$

$$\begin{aligned} \mathtt{Hom}_{\mathbf{C}}(T, R(\underset{\mathbf{I}}{\mathtt{lim}}\, D)) &\cong \mathtt{Hom}_{\mathbf{D}}(L(T), \underset{\mathbf{I}}{\mathtt{lim}}\, D) \\ &\cong \underset{\mathbf{I}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{D}}(L(T), D) \\ &\cong \underset{\mathbf{I}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{C}}(T, R(D)) \end{aligned}$$

- shows that $R(\mathtt{lim}_{\mathbf{I}} \mathbf{C}) \lhd R(C)$ is limit cone

– (must check that the isomorphism is induced by the correct map)

Argument for left adjoints similar: Übungsaufgabe

□

consider category $\mathbf{C}$

Definition 2.37. $\mathbf{C}$ *is called complete (cocomplete) if it admits limits (colimits) for all small diagrams*

Example: **Set** is complete and cocomplete

Example: consider the poset $\mathbb{N}$ as category

- consider diagram $X : \mathbf{I} \to \mathbb{N}$

– $\{X(i) \mid i \in \mathbf{I}\}$ is bounded iff $\mathtt{colim}_{\mathbf{I}} X$ exists

– in this case: $\mathtt{colim}_{\mathbf{I}} X = \max\{X(i) \mid i \in \mathbf{I}\}$

- $\mathbf{I}$ is not empty iff $\mathtt{lim}_{\mathbf{I}} X$ exists

– in this case $\mathtt{lim}_{\mathbf{I}} X = \min\{X(i) \mid i \in \mathbf{I}\}$

- so $\mathbb{N}$ is neither complete nor cocomplete if we add point ∞ larger then all other points: $\mathbb{N}_\infty := \{\infty\} \sqcup \mathbb{N}$ larger then all other points

- $\mathbb{N}_\infty$ is complete and cocomplete

Lemma 2.38.

1. *A small complete category* $\mathbf{C}$ *is cocomplete.*
2. *A small cocomplete category* $\mathbf{C}$ *is complete.*

Proof.

(1)

consider diagram $C : \mathbf{I} \to \mathbf{C}$

- use functor $\underline{(-)} : \mathbf{C} \to \mathbf{Fun}(\mathbf{I}, \mathbf{C})$ to define category of cones under C

$$\mathbf{Cone}(C/) := C \times_{\mathbf{Fun}(\mathbf{I},\mathbf{C})} \mathbf{Fun}(\mathbf{I}, \mathbf{C})_{C/}$$

- this category is also small and admits all limits
- $\mathtt{lim}_{\mathbf{Cone}(C/)}\, \mathtt{id}$ is an initial object of $\mathbf{Cone}(C/)$, hence a colimit cone under C

(2)

- consider $\mathbf{C}^{\mathrm{op}}$

□

consider the (non-small) category **Ord** of ordinals

- **Ord** is cocomplete: $\mathtt{colim}_{\mathbf{I}}\, X = \cup_{i \in \mathbf{I}} X(i)$
- **Ord** has all non-empty limits: $\mathtt{lim}_{\mathbf{I}}\, X = \cap_{i \in \mathbf{I}} X(i)$
- but **Ord** has no final object (hence not complete)

Lemma 2.39. *Die Kategorie* **Top** *ist vollständig und kovollständig.*

Proof. kovollständig:

- $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$ erhält alle Kolimiten, die in **Top** existieren
- $X : \mathbf{I} \to \mathbf{Top}$ - Diagramm
- Kandidat für underliegende Menge des Kolimes: $Y := \mathtt{colim}_{\mathbf{I}}\, \mathcal{F}(X)$
- $(c_i : \mathcal{F}(X(i)) \to Y)_{i \in \mathbf{I}}$ - Familie der kanonischen Abbildungen
- wähle für $\mathcal{T}_Y$ - größte Topologie so daß c_i für alle i in $\mathbf{I}$ stetig ist

Beh: $(c_i : X(i) \to (Y, \mathcal{T}_Y))_{i \in \mathbf{I}}$ definieren Kolimeskegel $X \triangleright (Y, \mathcal{T}_Y)$

- Z- topologischer Raum
- haben nach Konstruktion von Y Bijektion

$$\mathtt{Hom}_{\mathbf{Set}}(Y, \mathcal{F}(Z)) \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(X), \mathcal{F}(Z))$$

müssen zeigen: schränkt sich ein auf Bijektion

$$\mathtt{Hom}_{\mathbf{Top}}((Y, \mathcal{T}_Y), Z) \to \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Top}}(X, Z)$$

– ϕ in $\mathtt{Hom}_{\mathbf{Top}}((Y, \mathcal{T}_Y), Z)$ gegeben

– Bild ist $(\phi \circ c_i)_{i \in \mathbf{I}}$ - Familie stetiger Abbildungen, also in $\mathtt{lim}_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Top}}(X, Z)$

– ψ in $\mathtt{lim}_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Top}}(X, Z)$ gegeben

– korrespondiert erst mal zu ϕ in $\mathtt{Hom}_{\mathbf{Set}}(Y, \mathcal{F}(Z))$, z.z. ϕ ist stetig

– $\phi \circ c_i$ ist stetig für alle i in $\mathbf{I}$

– U in $\mathcal{T}_Z$

– $(\phi \circ c_i)^{-1}(U) = c_i^{-1}\phi^{-1}(U)$ is offen in Y für alle i

– also $\phi^{-1}(U)$ offen in Y

– schließen: ϕ ist stetig

haben damit Kovollständigkeit gezeigt

vollständig

- $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$ erhält alle Limiten, die in **Top** existieren
- $X : \mathbf{I} \to \mathbf{Top}$ - Diagramm
- Kandidat für underliegende Menge des Limes: $Y := \mathtt{lim}_{\mathbf{I}} \mathcal{F}(X)$
- $(e_i : Y \to \mathcal{F}(X(i)))_{i \in \mathbf{I}}$ - Familie der kanonischen Abbildungen
- wähle für $\mathcal{T}_Y$ - kleinste Topologie so daß e_i für alle i in $\mathbf{I}$ stetig ist

Beh: $(e_i : (Y, \mathcal{T}_Y) \to X(i))_{i \in \mathbf{I}}$ definieren Limeskegel $(Y, \mathcal{T}_Y) \triangleleft X$

- Z- topologischer Raum

– haben nach Konstruktion Bijektion

$$\mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(Z), \mathcal{F}(Y)) \xrightarrow{\cong} \lim_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(Z), \mathcal{F}(X))$$

- zu zeigen: schränkt sich ein zu

$$\mathtt{Hom}_{\mathbf{Top}}(Z,Y) \xrightarrow{\cong} \mathtt{lim}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Top}}(Z,X)$$

- ϕ in $\mathtt{Hom}_{\mathbf{Top}}(Z,Y)$ gegeben

– geht auf $(\phi \circ e_i)_{i \in \mathbf{I}}$ in $\mathtt{lim}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Top}}(Z,X)$ da $\phi \circ e_i$ für alle i in $\mathbf{I}$ stetig

- ψ in $\mathtt{lim}_{\mathbf{I}} \mathtt{Hom}_{\mathbf{Top}}(Z,X)$

– korrespondiert erst mal zu $\phi \in \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(Z), \mathcal{F}(Y))$, z.z. ϕ ist stetig

– $\psi = (e_i \circ \phi)_{i \in \mathbf{I}}$

– die Mengen der Form $e_i^{-1}(U)$ für i in $\mathbf{I}$ und U in $\mathcal{T}_{X(i)}$ erzeugen $\mathcal{T}_Y$

– $\phi^{-1}(e_i^{-1}(U)) = (e_i \circ \phi)^{-1}(U)$ is offen in Z da $e_i \circ \phi$ stetig ist

– schließen mit Lemma 2.14: ϕ ist stetig

□

2.3 Examples of limits and colimits in $\mathbf{Top}$

Unterräume:

X topologischer Raum, A Unterraum

Lemma 2.40. *The following square is cartesian:*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ A_{\text{chaot}} & \longrightarrow & X_{\text{chaot}} \end{array} .$$

Proof. verify using the description of the topology of A

$$\mathtt{Hom}_{\mathbf{Top}}(T,A) \cong \mathtt{Hom}_{\mathbf{Top}}(T,A_{\text{chaot}}) \times_{\mathtt{Hom}_{\mathbf{Top}}(T,X_{\text{chaot}})} \mathtt{Hom}_{\mathbf{Top}}(T,X)$$

- stetige Abbildungen nach sind stetige Abbildungen nach X, die Werte in A haben

– indeed: for $f : T \to A$ we have $f^{-1}(U \cap A) = f^{-1}(U)$

□

- can thus characterize the embedding of a subspace as a limit

G-invariants

G - a group

- BG - category with one object $*_{BG}$ and $\mathtt{Hom}_{BG}(*_{BG}, *_{BG}) \cong G$

– composition is multiplication in G

$\mathbf{C}$ a category

Definition 2.41. *The category* $\mathbf{Fun}(BG, \mathbf{C})$ *is the category of* G*-objects on* $\mathbf{C}$*.*

- objects: objects of C with an action of G
- morphisms: equivariant morphisms

- G-objects in **Top**
- $X : BG \to \mathbf{Top}$

– $X(*_{BG})$ - topological space

– G acts by continuous transformations on $X(*_{BG})$

Definition 2.42. *The space of* G*-fixed points in* X *is defined by*

$$X^G := \mathop{\mathtt{lim}}_{BG} X \ .$$

have canonical map $e : X^G \to X(*_{BG})$

Lemma 2.43. $e : X^G \to X(*_{BG})$ *is an inclusion of the subspace of points* x *in* X *with* $gx = x$ *for all* g *in* G*.*

Proof.

$(-)_{\mathrm{chaot}} : \mathbf{Set} \to \mathbf{Top}$ preserves limits

have inclusion of subsets (use explicit description of limits in **Set**)

$$\mathop{\mathtt{lim}}_{BG} \mathcal{F}(X) = \{x \in \mathcal{F}(X)(*_{BG}) \mid (\forall g \in X \mid gx = x)\} \to \mathcal{F}(X)(*_{BG})$$

define P by pull-back

$$\begin{array}{ccc} P & \xrightarrow{\quad e \quad} & X(*_{BG}) \\ \downarrow & & \downarrow \\ (\mathtt{lim}_{BG}\, \mathcal{F}(X))_{\mathrm{chaot}} & \longrightarrow & \mathcal{F}(X)(*_{BG})_{\mathrm{chaot}} \end{array}$$

- claim: $e : P \to X(*_{BG})$ defines limit cone $P \lhd X$

- to show: $\mathtt{Hom}_{\mathbf{Top}}(T, P) \to \mathtt{lim}_{BG} \mathtt{Hom}_{\mathbf{Top}}(T, X)$ (induced by e) is bijection

- indeed

$$\begin{aligned} \mathtt{Hom}_{\mathbf{Top}}(T, P) &\cong \mathtt{Hom}_{\mathbf{Top}}(T, (\mathop{\mathtt{lim}}_{BG} \mathcal{F}(X))_{\mathrm{chaot}}) \times_{\mathtt{Hom}_{\mathbf{Top}}(T, \mathcal{F}(X)(*_{BG})_{\mathrm{chaot}})} \mathtt{Hom}_{\mathbf{Top}}(T, X(*_{BG})) \\ &\cong \mathop{\mathtt{lim}}_{BG} \mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(T), \mathcal{F}(X)) \times_{\mathtt{Hom}_{\mathbf{Set}}(\mathcal{F}(T), \mathcal{F}(X)(*_{BG}))} \mathtt{Hom}_{\mathbf{Top}}(T, X(*_{BG})) \\ &\cong \mathop{\mathtt{lim}}_{BG} \mathtt{Hom}_{\mathbf{Top}}(T, X) \end{aligned}$$

- for the last isomorphism: $\mathtt{lim}_{BG} \mathtt{Hom}_{\mathbf{Top}}(T, X)$ is the set of continuous maps which are set-theoretically equivariant

- the last isomorphism expresses exactly this fact

the claim implies $X^G \cong P$

□

usually one writes also X instead of $X(*_{BG})$ for the underlying space of the G-space

X - a space with G-action

A - a subspace, G-invariant

Lemma 2.44. *$A^G \to X^G$ is an inclusion of subspaces.*

Proof. Is a special case of Lemma 2.45 below. □

$\mathbf{I}$ - small category

- $A, X : \mathbf{I} \to \mathbf{Top}$ diagrams

Lemma 2.45 (limits preserves inclusion of subspaces)**.** *If $A(i) \to X(i)$ is an inclusion of subspaces for every i in $\mathbf{I}$, then is $\mathtt{lim}_{\mathbf{I}} A \to \mathtt{lim}_{\mathbf{I}} X$ is an inclusion of subspaces.*

proof needs categorical preparation

- Limits commute with limits

- $\mathbf{I}$, $\mathbf{J}$ - small categories

- $\mathbf{C}$ category

Lemma 2.46 (Limits and colimits in functor categories are pointwise)**.**

1. *If $\mathbf{C}$ admits all $\mathbf{I}$-shaped limits, then so does $\mathbf{Fun}(\mathbf{J}, \mathbf{C})$ and for all C in $\mathbf{Fun}(\mathbf{I}, \mathbf{Fun}(\mathbf{J}, \mathbf{C}))$ and j in $\mathbf{J}$ we have*
$$(\mathop{\mathtt{lim}}_{\mathbf{I}} C)(j) \cong \mathop{\mathtt{lim}}_{\mathbf{I}} C(j, -) \ .$$

2. *If* $\mathbf{C}$ *admits all* $\mathbf{I}$*-shaped colimits, then so does* $\mathbf{Fun}(\mathbf{J},\mathbf{C})$ *and for all* C *in* $\mathbf{Fun}(\mathbf{I},\mathbf{Fun}(\mathbf{J},\mathbf{C}))$ *and* j *in* $\mathbf{J}$ *we have*

$$(\underset{\mathbf{I}}{\mathtt{colim}}\, C)(j) \cong \underset{\mathbf{I}}{\mathtt{colim}}\, C(j,-)\ .$$

Proof.

argument for (1)

use equivalences

$$\mathbf{Fun}(\mathbf{J},\mathbf{Fun}(\mathbf{I},\mathbf{C})) \simeq \mathbf{Fun}(\mathbf{I}\times\mathbf{J},\mathbf{C}) \simeq \mathbf{Fun}(\mathbf{J},\mathbf{Fun}(\mathbf{I},\mathbf{C}))$$

limit is a functor (as right-adjoint of $\underline{(-)}$)

- get functor $D:\mathbf{J}\to\mathbf{C}$, $\quad j\mapsto \mathtt{lim}_{\mathbf{I}}\, C(j,-)$ (pointwise application of limit)

- will see that it represents $\mathtt{lim}_{\mathbf{I}}\, C$ in $\mathbf{Fun}(\mathbf{J},\mathbf{C})$

- for arbitrary T in $\mathbf{Fun}(\mathbf{J},\mathbf{C})$:

$$\begin{aligned}\mathtt{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{Fun}(\mathbf{J},\mathbf{C}))}(\underline{T}_{\mathbf{I}},C) &\cong \mathtt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{Fun}(\mathbf{I},\mathbf{C}))}(\underline{T}^{pt}_{\mathbf{I}},C)\\ &\cong \mathtt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{C})}(T,D)\end{aligned}$$

– $\underline{T}^{pt}_{\mathbf{I}}$ - constant $\mathbf{I}$-diagram functor pointwise in j

- for second isomorphism:

- view $\mathtt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{D})}(A,B)\subseteq \prod_{j\in\mathbf{J}} \mathtt{Hom}_{\mathbf{D}}(A(j),B(j))$

- hence

$$\mathtt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{Fun}(\mathbf{I},\mathbf{C}))}(\underline{T}^{pt}_{\mathbf{I}},C)\subseteq \prod_j \mathtt{Hom}_{\mathbf{Fun}(\mathbf{I},\mathbf{C})}(\underline{T(j)},C(j,-))\cong \prod_j \mathtt{Hom}_{\mathbf{C}}(T(j),D(j))$$

– this subset is exactly $\mathtt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{C})}(T,D)$

for (2) analoguous □

Lemma 2.47 ((co)limits commute with (co)limits)**.**

1. *Assume that* $\mathbf{C}$ *admits all* $\mathbf{I}$ *and all* $\mathbf{J}$*-shaped limits. Then it admits* $\mathbf{I}\times\mathbf{J}$*-shaped limits and we have for every* C *in* $\mathbf{Fun}(\mathbf{J}\times\mathbf{I},\mathbf{C})$ *that*

$$\underset{\mathbf{J}}{\mathtt{lim}}\,\underset{\mathbf{I}}{\mathtt{lim}}\, C \cong \underset{\mathbf{J}\times\mathbf{I}}{\mathtt{lim}}\, C \cong \underset{\mathbf{I}}{\mathtt{lim}}\,\underset{\mathbf{J}}{\mathtt{lim}}\, \mathbf{C}\ .$$

2. *Assume that* $\mathbf{C}$ *admits all* $\mathbf{I}$ *and all* $\mathbf{J}$*-shaped colimits. Then it admits* $\mathbf{I} \times \mathbf{J}$*-shaped colimits and we have for every* C *in* $\mathbf{Fun}(\mathbf{J} \times \mathbf{I}, \mathbf{C})$ *that*

$$\operatorname*{\texttt{colim}}_{\mathbf{J}} \operatorname*{\texttt{colim}}_{\mathbf{I}} C \cong \operatorname*{\texttt{colim}}_{\mathbf{J}\times\mathbf{I}} C \cong \operatorname*{\texttt{colim}}_{\mathbf{I}} \operatorname*{\texttt{colim}}_{\mathbf{J}} \mathbf{C} \ .$$

Proof.

(1)

$C : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$ - a diagram

- T in $\mathbf{C}$ arbitrary

$$\begin{aligned} \texttt{Hom}_{\mathbf{C}}(T, \operatorname*{\texttt{lim}}_{\mathbf{J}} \operatorname*{\texttt{lim}}_{\mathbf{I}} C) &\cong \texttt{Hom}_{\mathbf{Fun}(\mathbf{J},\mathbf{C})}(\underline{T}_{\mathbf{J}}, \operatorname*{\texttt{lim}}_{\mathbf{I}} C) \\ &\cong \texttt{Hom}_{\mathbf{Fun}(\mathbf{J}\times\mathbf{I},\mathbf{C})}(\underline{T}_{\mathbf{J}\times\mathbf{I}}, C) \end{aligned}$$

shows that $\texttt{lim}_{\mathbf{J}\times\mathbf{I}} C$ exists and is isomorphic to $\texttt{lim}_{\mathbf{J}} \texttt{lim}_{\mathbf{I}} C$

(2)

analoguous

□

application:

Proof. (of Lemma 2.45) get diagram of the shape $\mathbf{I} \times (\bullet \to \bullet \leftarrow \bullet)$

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A_{\text{chaot}} & \longrightarrow & X_{\text{chaot}} \end{array} \quad .$$

by assumption for every i in $\mathbf{I}$ the following is cartesian

$$\begin{array}{ccc} A(i) & \longrightarrow & X(i) \\ \downarrow & & \downarrow \\ A(i)_{\text{chaot}} & \longrightarrow & X(i)_{\text{chaot}} \end{array} \quad .$$

hence (switch order of limits): the following is cartesian

$$\begin{array}{ccc} \mathtt{lim}_{\mathbf{I}} A & \longrightarrow & \mathtt{lim}_{\mathbf{I}} X \\ \downarrow & & \downarrow \\ \mathtt{lim}_{\mathbf{I}}(A_{\mathrm{chaot}}) & \longrightarrow & \mathtt{lim}_{\mathbf{I}}(X_{\mathrm{chaot}}) \end{array} .$$

- $(-)_{\mathrm{chaot}} : \mathbf{Top} \to \mathbf{Set} \to \mathbf{Top}$ is composition of right-adjoints and preserves limits: the following is cartesian

$$\begin{array}{ccc} \mathtt{lim}_{\mathbf{I}} A & \longrightarrow & \mathtt{lim}_{\mathbf{I}} X \\ \downarrow & & \downarrow \\ (\mathtt{lim}_{\mathbf{I}} A)_{\mathrm{chaot}} & \longrightarrow & (\mathtt{lim}_{\mathbf{I}} X)_{\mathrm{chaot}} \end{array} .$$

- $\mathtt{lim}_{\mathbf{I}}$ in sets preserves injective maps (by the explicit construction)

- hence upper arrow is inclusion of a subspace □

Quotienten:

$R \subseteq X \times X$ - Äquivalenzrelation

- statten R mit der Unterraumtopologie aus

- bilden coequalizer:

-

$$X/R := \mathtt{colim}\left(R \underset{\mathtt{pr}_1}{\overset{\mathtt{pr}_0}{\rightrightarrows}} X \right)$$

pictorial

$$R \underset{\mathtt{pr}_1}{\overset{\mathtt{pr}_0}{\rightrightarrows}} X \xrightarrow{\pi} X/R$$

– π is one of the canonical maps

– the other canonical map $R \to X$ is redundant

- explizit (nach allg. Konstruktion von Kolimits in **Top**):

– $\mathcal{F}(X/R)$ ist Menge der Äquivalenzklassen

– $U \in \mathcal{P}_{X/R}$ offen genau dann wenn $\pi^{-1}(U)$ offen

- universelle Eigenschaft:

$$\mathtt{Hom}_{\mathbf{Top}}(X/R, T) \cong \mathtt{Hom}^R(X; T)$$

($\mathtt{Hom}^R(-,-)$ - Abbildungen, die Äquivalenzklassen auf Punkte abbilden)

Lemma 2.48. *Assume:*

1. *$X \to X/R$ is open.*
2. *X is Hausdorff.*
3. *R is closed.*

Then X/R is Hausdorff.

Proof. consider $[x], [x']$ - points in X/R such that $[x] \neq [x']$

- then (x, x') in $(X \times X) \setminus R$
- $(X \times X) \setminus R$ is open
- find open neighbourhoods U of x and U' of x' with $U \times U' \subseteq (X \times X) \setminus R$
- then: $\pi(U)$ and $\pi(U')$ are opens in X/R separating $[x]$ and $[x']$ and $\pi(U) \cap \pi(U') = \emptyset$ □

C - category with finite products

- admits final object $*$ (empty product)

Definition 2.49. *A group object in* **C** *is a triple (G, μ, e) of*

1. *an object G in* **C***,*
2. *a morphism $\mu : G \times G \to G$,*
3. *a morphism $e : * \to G$,*

satisfying the conditions

1. *associativity: $\mu \circ (\mu \times \mathtt{id}_G) = \mu \circ (\mathtt{id}_G \times \mu) : G \times G \times G \to G$,*
2. *unit: $\mu \times (e \times \mathtt{id}_G) = \mathtt{id}_G$, $\mu \times (\mathtt{id}_G \times e) = \mathtt{id}_G$,*
3. *shear map: the map $(\mathtt{pr}, \mu) : G \times G \to G \times G$ is an isomorphism.*

Definition 2.50. *A topological group is a group object in* **Top**.

- in detail:

– a group G with a topology such that

– multiplication $G \times G \to G$ is continuous

– shear map $G \times G \to G \times G$, $(g,h) \mapsto (g, gh)$ is a homeomorphism

G - topological group

X - topological pace

Definition 2.51. *G acts continuously on X if $\mathcal{F}(G)$ acts on $\mathcal{F}(X)$ and the action map $a : G \times X \to X$ is continuous.*

let G act continuously on X

- $a : G \times X \to X$ - action map
- $p : G \times X \to X$ - projection

define quotient of X by G as coequalizer

$$X/G := \mathtt{colim}\left(G \times X \underset{p}{\overset{a}{\rightrightarrows}} X \right)$$

pictorially

$$G \times X \underset{p}{\overset{a}{\rightrightarrows}} X \xrightarrow{\pi} X/G$$

Example:

- G is a group
- G_{disc} is a topological group
- consider X in $\mathbf{Fun}(BG, \mathbf{Top})$

– then G_{disc} acts continuously on X

Lemma 2.52. *We have an isomorphism $X/G_{\mathrm{disc}} \cong \mathtt{colim}_{BG} X$.*

Proof. Exercise. □

define subset $R_G := (a, p)(G \times X)$ of $X \times X$

Lemma 2.53. *We have an isomorphism*

$$X/G \cong X/R_G \ .$$

Proof. Exercise. □

X - a topological space

G - a topological group

- G acts continuously on X

Lemma 2.54. *The map $X \to X/G$ is open.*

Proof.

consider U - open in X

- $U' := \bigcup_{g \in G} gU$ is open
- $\pi^{-1}(\pi(U)) = U'$ is open
- hence $\pi(U)$ is open □

$f : X \to Y$ - morphism in **Top**

Definition 2.55. *f is called proper if for every compact subset K of Y the subset $f^{-1}(K)$ is compact.*

Definition 2.56. *G acts properly if $(a, p) : G \times X \to X \times X$ is a proper map.*

X topological space

Definition 2.57. *X ist lokal-kompakt, wenn jeder Punkt in y eine kompakte Umgebung besitzt.*

Lemma 2.58. *Let X be locally compact. Then for every point x in X and open neighbourhood V of x there exists a compact neighbourhood K such that $K \subseteq V$.*

in other words: for every point in X the poset of its neighbourhoods admits a cofinal set of compact neighbourhoods

Proof.

choose a compact neighbourhood L of x

- L is Hausdorff (as a compact space)

– for every ℓ in $L \setminus V$ we have $\ell \neq x$

— choose an open nbhd V_ℓ of ℓ in L and open W_ℓ of x in X such that $V_\ell \cap W_\ell = \emptyset$

—- we can choose W_ℓ open in X since L is a neighbourhood of x

—– first choose open in W'_ℓ in X with $x \in W'_\ell$ and $W'_\ell \cap V_\ell = \emptyset$ (possible since L is Hausdorff)

—– chose open W''_ℓ in X such that $W'_\ell = L \cap W''_\ell$ (since L has subspace topology).

—– let W_ℓ be intersection of W''_ℓ with some open nbhd of x contained in L (the latter exists since L is a nbhd of x)

- $L \setminus V$ is closed in L (intersection of the closed subset $X \setminus V$ with L) and hence quasi compact

– there exists a finite set I of $L \setminus V$ such that $L \setminus V \subseteq \bigcup_{\ell \in I} V_\ell$

– $K := \bigcap_{\ell \in I} L \setminus V_\ell = L \setminus \bigcup_{\ell \in I} V_\ell$ is closed in L and hence compact

– $x \in \bigcap_{\ell \in I} W_\ell \subseteq K \subseteq V$ shows:

— K is a neighbourhood of x and contained in V

□

W - subset of topological space X

Definition 2.59. *W is locally closed if there exists an open subset U and a closed subset A of X such that $W = A \cap U$.*

Examples:

$\mathbb{R}^n$ is locally compact

Lemma 2.60. *Locally closed subsets of locally compact spaces are locally compact.*

Proof.

W in X locally closed

- $W = U \cap A$ for open U and closed A

w in W

- K compact neighbourhood of w in X contained in U

– exists by Lemma 2.58

- $K \cap A = K \cap W$ is compact neighbourhood of w in W

□

locally closed subsets of $\mathbb{R}^n$ are locally compact

Example:

- $X := \mathbb{R} \sqcup_{(-\infty,0)} \mathbb{R}$ is locally compact, but not Hausdorff

– X is locally homomorphic to $\mathbb{R}$

– hence X is locally compact

– in X there are compacts which are not closed: e.g. $[-1,1] \sqcup \emptyset$

X - topological space

Lemma 2.61. *Assume:*

1. *$X \to X/R$ is open.*
2. *X is Hausdorff.*
3. *R is closed.*
4. *X locally compact.*

Then X/R is locally compact and Hausdorff.

Proof.

Hausdorff by Lemma 2.48

$[x]$ in X/G

– K_x - compact neighbourhood of x

— $\pi(K_x)$ is compact since X/R is Hausdorff

— $\pi(K_x)$ is neighbouhood of x (since π is open)

- $\pi(K_x)$ is compact neighbouhood of $[x]$

□

Lemma 2.62. *Assume:*

1. *X is Hausdorff and locally compact*
2. *G acts properly on X.*

Then X/G is locally compact and Hausdorff.

Proof.

$\pi : X \to X/G$ is open by Lemma 2.54

- claim: R_G is closed in $X \times X$

– show that $(X \times X) \setminus R_G$ is open

– (x, x') in $(X \times X) \setminus R_G$

– K_x - compact neighbourhood of x, $K_{x'}$ compact neighbourhood of x'

– $R_G \cap (K_x \times K_x)$ is image of compact set in a Hausdorff space and hence closed

– find neighbourhoods U_x of x in K_x and $U_{x'}$ of x' in $K_{x'}$ such that $(U_x \times U_{x'}) \cap R_G = \emptyset$

- apply Lemma 2.61

□

Examples:

if G is compact, then it acts properly on any space

$\mathbb{Z}$ acts properly on $\mathbb{R}$ by $(n, r) \mapsto n + r$

- preimage of $[-n, n] \times [-n, n]$ in $\mathbb{Z}_{\text{disc}} \times \mathbb{R}$ is contained in $[-2n, 2n] \times [-n, n]$

- $\mathbb{R}/\mathbb{Z} \cong S^1$

- isomorphism given by $[t] \mapsto e^{2\pi i t}$

- Verification: Exercise

$\mathbb{Q}$ acts on $\mathbb{R}$ by $(q, r) \mapsto q + r$

- $\mathbb{R}/\mathbb{Q}$ has the chaotic topology.

– every open subset of $\mathbb{R}/\mathbb{Q}$ is of the form $\pi(U)$ for some open U in $\mathbb{R}$

– of $U \neq \emptyset$, then $\mathbb{Q} + U = \mathbb{R}$ since $\mathbb{Q}$ dense in $\mathbb{R}$

– $\mathbb{R}/\mathbb{Q} = \pi(\mathbb{R}) = \pi(\mathbb{Q} + U) = \pi(U)$

– action not proper

More examples:

Es gibt ein Push-out Diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n \end{array}$$

Es gibt ein Push-out Diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathbb{RP}^{n-1} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & \mathbb{RP}^n \end{array}$$

Es gibt ein Push-out Diagram

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & \mathbb{CP}^{n-1} \\ \downarrow & & \downarrow \\ D^{2n} & \longrightarrow & \mathbb{CP}^n \end{array}$$

2.4 Mapping spaces

X, Y - topologisch

- have set $\mathtt{Hom}_{\mathbf{Top}}(X, Y)$
- equip it with the compact-open topology

Definition 2.63. *The compact-open topology on* $\mathtt{Hom}_{\mathbf{Top}}(X, Y)$ *is generated by the sets*

$$W(K, U) := \{f \in \mathtt{Hom}_{\mathbf{Top}}(X, Y) \mid f(K) \subseteq U\}$$

for all compact subsets K *of* X *and open subsets* U *of* Y*.*

schreiben $\mathtt{Map}(X, Y)$ für diesen topologischen Raum

Example:

X - a set, Y a space

- have homeomorphism $\mathtt{Map}(X_{\text{disc}}, Y) \cong \prod_{x \in X} Y$
- $f \mapsto (f(x))_{x \in X}$

X, Y, Z - topological spaces,

- get composition

$$\circ : \mathtt{Map}(Y, Z) \times \mathtt{Map}(X, Y) \to \mathtt{Map}(X, Z)$$

Lemma 2.64. *If* Y *is Hausdorff and locally compact, then the composition is continuous.*

Proof.

check on generators

$W(K,U)$ in $\texttt{Map}(X,Z)$

(f,g) in $\circ^{-1}(\texttt{Map}(X,Z))$

- $(f \circ g)(K) \subseteq U$
- $g(K)$ compact in Y (since Y is Hausdorff)
- $g(K) \subseteq f^{-1}(U)$
- find open V in Y with

– $g(K) \subseteq V \subseteq f^{-1}(U)$ and $\bar{V} \subseteq f^{-1}(U)$ compact

— to this end: use here that Y is locally compact and Lemma 2.58

— every point of $g(K)$ admits compact neighbourhood contained in $f^{-1}(U)$

—- cover $g(K)$ by the interiors finitely many of those compact neigbourhoods

—- define V as the union of these interiors

—- $\bar{V}$ is the union of these neighbourhoods

- $\circ(W(\bar{V},U) \times W(K,V)) \subseteq W(K,U)$

– indeed: g' in $W(K,V)$ and f' in $W(\bar{V},U)$

– $(f' \circ g')(K) \subseteq f'(g'(K)) \subseteq f'(V) \subseteq f'(\bar{V}) \subseteq U$

□

example

X a topological space

- $\texttt{Aut}(X) \subseteq \texttt{Hom}_{\mathbf{Top}}(X,X)$ is group

X - locally compact

- $\texttt{Aut}(X)$ becomes topological monoid

– composition is continuous

– not clear that shear map is iso

$G \subseteq \texttt{Aut}$ compact submonoid

- closed under taking inverses
- then G is a topological group

- shear map $G \times G \to G \times G$ is a bijection between compact sets, hence an isomorphism (Lemma 2.16)

exponential law:

- für X, YZ in **Set**

$$\mathtt{Hom}_{\mathbf{Set}}(X \times Y, Z) \cong \mathtt{Hom}_{\mathbf{Set}}(X, \mathtt{Hom}_{\mathbf{Set}}(Y, Z))$$

X, Y, Z in **Top**

Lemma 2.65. *Wenn Y lokal-kompakt ist, dann gilt das Exponentialgesetz*

$$\mathtt{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(Y, Z)) \ .$$

Proof. Bijektion $\phi \leftrightarrows \psi$

start with bijection

$$\mathtt{Hom}_{\mathbf{Set}}(X \times Y, Z) \cong \mathtt{Hom}_{\mathbf{Set}}(X, \mathtt{Hom}_{\mathbf{Set}}(Y, Z))$$

$$\phi \leftrightarrows \psi$$

Annahme: $\phi \in \mathtt{Hom}_{\mathbf{Top}}(X \times Y, Z)$

- zu zeigen: ψ in $\mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(Y, Z))$ wohldefiniert

– x in X

— conclude $\psi(x) \in \mathtt{Hom}_{\mathbf{Top}}(Y, Z)$ als Komposition $Y \stackrel{y \mapsto (x,y)}{\to} X \times Y \stackrel{\phi}{\to} Z$

– show now that ψ is continuous

— fix generator $W(K, U)$ of topology

— x in $\psi^{-1}(W(K, U))$ besagt $\phi(x, K) \subseteq U$

— für every y in K existieren Umgebungen U_y von y und $U_{y,x}$ von x mit $\phi(U_{y,x} \times U_y) \subseteq U$

— since K is compact there exists finite subset $I \subseteq K$ such that

— $K \subseteq \bigcup_{y \in I} U_y$

— $V := \bigcap_{i \in I} U_{y,x}$ is offene Umgebung von x

— $\phi(V \times K) \subseteq U$

— consequently $V \subseteq \psi^{-1}(W(K, U))$

– final conclusion: $\psi^{-1}(W(K,U))$ offen

Annahme: ψ in $\mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(Y,Z))$

- must show that $\phi \in \mathtt{Hom}_{\mathbf{Top}}(X \times Y, Z)$

– (x,y) in $\phi^{-1}(U)$

– K kompakte Umgebung von y

– $V := \psi^{-1}(W(K,U))$ offen, $x \in V$

– $V \times K \subseteq \phi^{-1}(U)$

– conclude $\phi^{-1}(U)$ is open

□

Corollary 2.66. *Wenn Y lokal-kompakt ist, dann gilt:*

1. *Es gibt es eine Adjunktion*

$$- \times Y : \mathbf{Top} \leftrightarrows \mathbf{Top} : \mathtt{Map}(Y,-) \ .$$

2. *$- \times Y$ erhält Kolimiten.*
3. *$\mathtt{Map}(Y,-)$ erhält Limiten.*

Example:

X - a G-space, Y locally compact

- $(X/G) \times Y \cong (X \times Y)/G$

- $\mathtt{Map}(Y, X^G) \cong \mathtt{Map}(Y,X)^G$

if Z is locally compact, the $Z \times - : \mathbf{Top} \to \mathbf{Top}$ preserves all colimits

- for general Z in **Top**: $Z \times -$ preserves some colimits

Lemma 2.67.

1. *The functor $Z \times - : \mathbf{Top} \to \mathbf{Top}$ preserves open maps.*
2. *The functor $Z \times - : \mathbf{Top} \to \mathbf{Top}$ preserves embeddings of closed subsets*

Proof.

(1)

assume: $f : X \to Y$ open

- U open in $Z \times X$

- for every (z, x) choose open neighbourhoods V_z of z in Z and W_x of x in X such that $V_z \times W_x \subseteq U$

- then $U = \bigcup_{(z,x)\in U} V_z \times W_x$

- $(Z \times f)(U) = (Z \times f)(\bigcup_{(z,x)\in U} V_z \times W_x) = \bigcup_{(z,x)\in U} V_z \times f(W_x)$

- for every (z, x) the set $V_z \times f(W_x)$ is open in $Z \times Y$

- $(Z \times f)(U)$ is open in $Z \times Y$

(2)

consider closed subset A in X

- then $Z \times A$ is subset of $Z \times X$

- must show: is also closed

- complement is $Z \times (X \setminus A)$ - this is open in $Z \times X$ since $X \setminus A$ is open in X □

consider diagram $X : \mathbf{I} \to \mathbf{Top}$

Z in **Top**

- get canonical map $\mathtt{colim}_{\mathbf{I}}(Z \times X) \to Z \times \mathtt{colim}_{\mathbf{I}} X$.

Lemma 2.68. *The canonical map* $\mathtt{colim}_{\mathbf{I}}(Z \times X) \to Z \times \mathtt{colim}_{\mathbf{I}} X$ *is an isomorphisms in the following cases:*

1. $\mathbf{I}$ *is discrete.*
2. $\coprod_{i\in\mathbf{I}} X(i) \to \mathtt{colim}_{\mathbf{I}} X$ *is open*
3. $\mathbf{I}$ *is finite and for every* i *the canonical map* $c_i : X(i) \to \mathtt{colim}_{\mathbf{I}} X$ *is an embedding of a closed subset.*

Proof.

in **Set**:

- the functor $A \times - : \mathbf{Set} \to \mathbf{Set}$ is a left adjoint (with right adjoint $\mathtt{Hom}_{\mathbf{Set}}(A, -)$)

- $A \times -$ preserves colimits in **Set**

- apply forgetful functor $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$ which commutes with colimits

– $\mathtt{colim}_{\mathbf{I}}(Z \times X) \to Z \times \mathtt{colim}_{\mathbf{I}} X$ is a bijection of the underlying sets

- it remains to show that it is open or closed

(1)

- show that canonical map is open

- write $\mathtt{colim}_{\mathbf{I}} = \coprod_{\mathbf{I}}$

- let U be open in $\coprod_{i \in \mathbf{I}}(Z \times X(i))$

- must show: U is open in $Z \times \coprod_{i \in \mathbf{I}} X(i)$

– consider (z, x) in U

– show that there exists open subset U' of $Z \times \coprod_{i \in \mathbf{I}} X(i)$ with $(z, x) \subseteq U' \subseteq U$

- then conclusion: U is also open in $Z \times \coprod_{i \in \mathbf{I}} X(i)$

— assume $x \in X(j)$ for j in $\mathbf{I}$

—- $U \cap (Z \times X(j))$ is open

—- find neighbourhoods V of z in Z and W of x in $X(j)$ such that $V \times W \subseteq U \cap (Z \times X(j))$

—- W is open in $\coprod_{i \in \mathbf{I}} X(i)$

— $U' := V \times W$ is open neighbourhood of (z, x) in $Z \times \coprod_{i \in \mathbf{I}} X(i)$ and $U' \subseteq U$

(2)

show that canonical map is open

$$\begin{array}{ccc} Z \times \coprod_{i \in \mathbf{I}} X(i) & \xrightarrow{\text{open}} & Z \times \mathtt{colim}_{\mathbf{I}} X \\ \uparrow \cong & & ? \uparrow \\ \coprod_{i \in \mathbf{I}}(Z \times X(i)) & \longrightarrow & \mathtt{colim}_{\mathbf{I}}(Z \times X) \end{array}$$

- upper horizontal map is open by assumption and Lemma 2.67

- left vertical map is iso by (1)

- consider U in $\mathtt{colim}_{\mathbf{I}}(Z \times X)$ open

- there exists open $\tilde{U}$ in $\coprod_{i \in \mathbf{I}}(Z \times X(i))$ mapping to U

- image of $\tilde{U}$ in $Z \times \mathtt{colim}_{\mathbf{I}} X$ is open (go up and right) and is also image of U under ?

(3)

show that canonical map is closed

- A in $\mathtt{colim}_{\mathbf{I}}(Z \times X(i))$ closed
- $\tilde{c}_i : Z \times X(i) \to \mathtt{colim}_{\mathbf{I}}(Z \times X(i))$ canonical map
- $\tilde{c}_i = \mathtt{id}_Z \times c_i$ for canonical map $c_i : X(i) \to \mathtt{colim}_{\mathbf{I}} X$
- $\tilde{c}_i = \mathtt{id}_Z \times c_i : Z \times X(i) \to Z \times \mathtt{colim}_{\mathbf{I}} X$ is embedding of closed subset (by assumption and Lemma 2.67)
- $\tilde{c}_i^{-1}(A)$ closed in $Z \times X(i)$ for every i in $\mathbf{I}$ (since $\tilde{c}_i$ is continuous)

– $A \bigcap (Z \times c_i(X(i)))$ is closed in $\tilde{c}_i(Z \times X(i))$ (since $\tilde{c}_i$ is an embedding)

– $A \bigcap (Z \times c_i(X(i)))$ is closed in $Z \times \mathtt{colim}_{\mathbf{I}} X$ (since $\tilde{c}_i(Z \times X(i))$ is a closed subset)

- $A = \bigcup_{i \in \mathbf{I}}(A \cap (Z \times c_i(X(i))))$ is closed in $Z \times \mathtt{colim}_{\mathbf{I}} X$ (since $\mathbf{I}$ is finite)

□

examples:

$Z \times -$ preserves coproducts

Corollary 2.69. *$Z \times -$ preserves quotients by group actions.*

Proof.

X - a G-space

write X/G as coequalizer of $a, \mathtt{pr}_X : G_{\text{disc}} \times X \to X/G$

- show that $G_{\text{disc}} \times X \sqcup X \to X/G$ is open and apply Lemma 2.68 (2)

— $X \to X/G$ is open

— $G_{\text{disc}} \times X \to X/G$ is the map $G_{\text{disc}} \times X \stackrel{\mathtt{pr}_X}{\to} X \to X/G$

— is composition of open maps

□

for Z in **Top** and G-space X we have $(Z \times X)/G \cong Z \times X/G$

$Y : \mathbf{I} \to \mathbf{Top}$

Lemma 2.70. *Assume:*

1. *$Y(i)$ is locally compact for every i in $\mathbf{I}$.*
2. *$\mathtt{colim}_{\mathbf{I}} Y$ is a locally compact space.*
3. *For every X in **Top** the canonical map $\mathtt{colim}_{\mathbf{I}}(X \times Y) \to X \times \mathtt{colim}_{\mathbf{I}} Y$ is an isomorphism.*

*Then for every Z in **Top** we have a canonical isomorphism*

$$\mathtt{Map}(\mathtt{colim}_{\mathbf{I}} Y, Z) \cong \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Map}(Y, Z) \ .$$

we have an isomorphism.

Proof.

X arbitrary, have natural isomorphism

$$\begin{aligned}
\mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(\mathtt{colim}_{\mathbf{I}} Y, Z)) &\cong \mathtt{Hom}_{\mathbf{Top}}(X \times \mathtt{colim}_{\mathbf{I}} Y, Z) \\
&\cong \mathtt{Hom}_{\mathbf{Top}}(\mathtt{colim}_{\mathbf{I}}(X \times Y), Z) \\
&\cong \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Top}}(X \times Y, Z) \\
&\cong \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(Y, Z)) \\
&\cong \mathtt{Hom}_{\mathbf{Top}}(X, \lim_{\mathbf{I}^{\mathrm{op}}} \mathtt{Map}(Y, Z))
\end{aligned}$$

□

Examples:

G a topological group, acts properly on locally compact Hausdorff space Y

- then $\mathtt{Map}(Y^G, Z) \cong \mathtt{Map}(Y, Z)^G \cong \mathtt{lim}\,(\mathtt{Map}(Y, Z) \rightrightarrows \mathtt{Map}(G \times Y, Z))$
- use Cor. 2.69 and Lemma 2.62
- the equalizer consists of all maps $f : Y \to Z$ with $f(gy) = f(y)$ for all g in G

– this is exactly the fixed point set

loop space:

- $S^1 \cong [0, 1] \cup_{1=2,0=1} [1, 2] = \mathtt{colim}([0, 1] \leftarrow \{0, 1\} \to [1, 2])$
- $\mathtt{Map}(S^1, Z) \cong \mathtt{Map}([0, 1], Z) \times_{Z \times Z} \mathtt{Map}([1, 2], Z)$
- use: $[0, 1], \{0, 1\}, [1, 2]$ are closed subspaces of S^1
- use Lemma 2.68

2.5 Homotopie und die Homotopiekategorie

$f_0, f_1 : X \to Y$ Morphismen in **Top**

Definition 2.71. *f_0 und f_1 heißen zueinander homotop wenn es eine Abbildung $H : [0,1] \times X \to Y$ (eine Homotopie) gibt mit $f_i = H_{|\{i\} \times X} : X \cong \{i\} \times X \to Y$.*

- equivalently: H is a map $X \to \mathtt{Map}([0,1], Y)$

– $X \to \mathtt{Map}([0,1], Y) \stackrel{\mathtt{ev}_i}{\to} Y$ is f_i

Notation:

- write $f_0 \sim f_1$ or $f_0 \stackrel{H}{\sim} f_1$

Lemma 2.72.

1. *Homotopie ist eine Äquivalenzrelation.*
2. *Homotopie ist kompatibel mit der Komposition in* **Top**.

Proof.

(1)

reflexivity:

- $f : X \to Y$

– $f \stackrel{H}{\sim} f$ for

– $H : [0,1] \times X \stackrel{\mathtt{pr}}{\to} X \stackrel{f}{\to} Y$

symmetry

- $f_0, f_1 : X \to Y$

– $f_0 \stackrel{H}{\sim} f_1$ implies $f_1 \stackrel{H'}{\sim} f_0$ with

– $H' : [0,1] \times X \stackrel{(t,x) \mapsto (1-t,x)}{\to} [0,1] \times X \stackrel{H}{\to} Y$

Idea: composition of homotopies

$[0,2] \cong [0,1] \sqcup_{\{1\}} [1,2]$

- $H', H : [0,1] \times X \to Y$

- assume $H(1,-) = H'(0,-)$

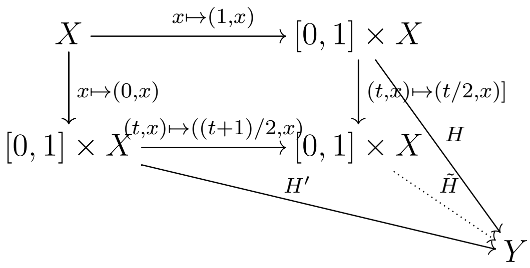
- set $H'' : [1,2] \times X \to Y$, $H''(s,x) := H'(s-1,x)$

transitivity:

- $f_0, f_1, f_2 : X \to Y$

– $f_0 \overset{H}{\sim} f_1$, $f_1 \overset{H'}{\sim} f_2$

– consider diagram



- use Lemma 2.68, (3) to see that square is a pushout

– get $\tilde{H}$ from universal property

- conclude: $f_0 \overset{\tilde{H}}{\sim} f_2$

(2)

$f_0, f_1 : X \to Y$, $g : Y \to Z$, $h : W \to X$

- $f_0 \overset{H}{\sim} f_1$ implies $g \circ f_0 \overset{g \times H}{\sim} g \circ f_1$ and $f_0 \circ h \overset{H \circ (\mathtt{id} \times h)}{\sim} f_1 \circ h$

□

Bilden Homotopiekategorie **hTop**

- Objekte: topologische Räume

- Morphismen: Homotopieklassen von Morphismen, Komposition induziert

– $[f] \circ [g] := [f \circ g]$

– is well-defined by Lemma 2.72

Top $\to$ **hTop** kanonischer Funktor, $X \mapsto X$, $f \mapsto [f]$

X, Y in **Top**

$f : X \to Y$

Definition 2.73.

1. *f ist eine Homotopieäquivaenz, wenn $[f] : X \to Y$ in* **hTop** *ein Isomorphismus ist.*

2. X, Y sind homotopieäquivalent, wenn sie isomorph in **hTop** *sind.*

schreiben $X \simeq Y$ für die Relation "Homotopieäquivalenz"

wenn $[g] = [f]^{-1}$ ist, dann heißt g ein Homotopieinverses von f

Beispiel:

$i : \{0\} \to D^n$ ist Homotopieäquivalenz

- Homotopieinverse: $\pi : D^n \to \{0\}$

- $\pi \circ i = \mathtt{id}$

- $i \circ \pi \overset{H}{\sim} \mathtt{id}$

– mit $H : [0,1] \times D^n \to D^n$

– $H(s,x) := sx$

— $H(1,-) = \mathtt{id}$

— $H(0,-) = i \circ \pi$

$i : \{0\} \to \mathbb{R}^n$ ist Homotopieäquivalenz

- Homotopieinverse: $\pi : \mathbb{R}^n \to \{0\}$

- $\pi \circ i = \mathtt{id}$

- $i \circ \pi \overset{H}{\sim} \mathtt{id}$

– mit $H : [0,1] \times R^n \to R^n$

– $H(s,x) := sx$

— $H(1,-) = \mathtt{id}$

— $H(0,-) = i \circ \pi$

- $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ ist eine Homotopieäquivalenz

– Homotopieinverse: $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $x \mapsto \frac{x}{\|x\|}$

– $\pi \circ i = \mathtt{id}$

- $i \circ \pi \overset{H}{\sim} \mathtt{id}$

– mit $H : [0,1] \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$

– $H(t,x) := tx + (1-t)\frac{x}{\|x\|}$

– $H(1,-) = \mathtt{id}$

– $H(0,-) = i \circ \pi$

- $\pi : [0,1] \times X \to X$ ist eine Homotopieäquivalenz

- Homotopieinverse $i_0 : X \to [0,1] \times X$, $x \mapsto (0,x)$

- $\pi \circ i_0 = \mathtt{id}$

- $i_0 \circ \pi \overset{H}{\sim} \mathtt{id}$

– mit $H : [0,1] \times ([0,1] \times X) \to [0,1] \times X$,

– $H(s,(t,x)) := (st, x)$

Bemerkung:

$i_t : X \to [0,1] \times X$, $x \mapsto (t,x)$ ist auch ein Homotopieinverses für jedes t in $[0,1]$

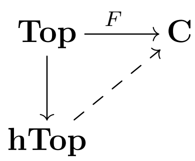
C - Kategorie

F : **Top** $\to$ **C** - Funktor

Definition 2.74. *F ist homotopieinvariant wenn für alle Paare homotoper Morphimsmen $f_0, f_1 : X \to Y$ in* **Top** *gilt $F(f_0) = F(f_1)$.*

Lemma 2.75. *Die folgenden Aussagen sind äquivalent:*

1. *F is homotopieinvariant.*
2. *F faktorisiert über* **Top** $\to$ **hTop**.



3. *Für alle X in* **Top** *induziert die Projektion $[0,1] \times X \to X$ einen Isomorphismus $F([0,1] \times X) \to F(X)$.*

Proof. (1) $\Rightarrow$ (2)

- ist klar

(2) $\Rightarrow$ (3)

- wenn $[f]$ iso in **hTop**, dann $F(f)$ iso in **C**

- $[0,1] \times X \to X$ ist iso in **hTop**, also $F([0,1] \times X) \to F(X)$ ist iso

(3) $\Rightarrow$ (1)

- $F(i_i) : F(X) \to F([0,1] \times X)$ Inverse zu $F([0,1] \times X) \to F(X)$ (da $p \circ i_i = \mathtt{id}_X$) und damit gleich

- seien nun f_0, f_1 homotop mit Homotopie H

- $F(f_0) = F(H \circ i_0) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(H \circ i_1) = F(f_1)$

□

2.6 π_0 als Beispiel eines homotopieinvarianten Funktors

X topologischer Raum

Definition 2.76. *X heißt zusammenhängend, wenn A keine nicht-triviale Zerlegung in zwei disjunkte offene Teilmengen besitzt.*

- $\mathbb{R}^n$ ist zusammenhängend
- S^{n-1} ist zusammenhängend
- $C_{1/3}$ ist nicht zusammenhängend:
  – $C_{1/3} \cap [0, 1/2)$ und $C_{1/3} \cap (1/2, 1]$ bilden disjunkte offene Zerlegung

$f : X \to Y$ Morphismus

Lemma 2.77. *Wenn X zusammenhängend ist, dann ist $f(X)$ zusammenhängend.*

Proof. Annahme: $f(X)$ nicht zusammenhängend

- $f(X) = U \cup V$ mit U, V offen in $f(X)$, disjunkt
- $X = f^{-1}(U) \cup f^{-1}(V)$ offene disjunkte Zerlegung von X
- also X nicht zusammenhängend

□

Topologischer Raum

x in X

Definition 2.78. *Die Zusammenhangskomponente von x ist durch*

$$[x] := \bigcup_{x \in A \subseteq X, A \text{ zush.}} A$$

definiert.

Lemma 2.79.

1. *$[x]$ ist zusammenhängend.*

2. *$[x]$ ist abgeschlossen.*

3. *Für x, y in X gilt entweder $[x] = [y]$ oder $[x] \cap [y] = \emptyset$.*

Proof. (1)

- $[x] = U \cup V$ disjunkte offene Zerlegung, obda $x \in U$
- für jedes zusammenhängende A mit $x \in A$ gilt $V \cap A = \emptyset$
– (sonst wäre $(A \cap U, A \cap V)$ eine nichttriviale disjunkte offene Zerlegung)
– also $V \cap [x] = \emptyset$
– also $V = \emptyset$

(2)

- zeigen: $\overline{[x]}$ ist zusammenhängend
– daraus folgt $\overline{[x]} \subseteq [x]$
– wegen $[x] \subseteq \overline{[x]}$ gilt dann $[x] = \overline{[x]}$
- $\overline{[x]} = U \cup V$ disjunkte Zerlegung mit U, V offen
– dann sind U, V auch abgeschlossen in $\overline{[x]}$
- obda $x \in U$
— dann $[x] \subseteq U$
- also $\overline{[x]} \subseteq U$ (da U abgeschlossen)
- also $V = \emptyset$

(3)

- $z \in [y] \Rightarrow [y] \subseteq [z] \Rightarrow y \in [z] \Rightarrow [z] \subseteq [y]$
- also wenn $[x] \cap [y] \neq \emptyset$ wähle $z \in [x] \cap [y]$
– dann $[z] = [y]$ und $[z] = [x]$

□

haben Äquivalenzrelation auf X: $x \sim y := [x] = [y]$

- equivalence classes are the connected components

Definition 2.80. *$\pi_0(X)$ ist die Menge der Zusammenhangskomponenten von X.*

X - toplogischer Raum

Definition 2.81. *X heißt total unzusammenhängend, wenn die Zusammenhangskomponenten von X Punkte sind.*

- X is totally disconnected if and only if $X \to \pi_0(X)$ is a bijection

Beispiel:

für Menge X ist X_{disc} total unzusammenhängend (und Punkte sind offen)

$C_{1/3}$ ist total unzusammenhängend, Punkte aber nicht offen

$\pi_0(X)$ hat Quotiententopologie

- es gilt $[[x]] = \{[x]\}$
- die Punkte in $\pi_0(X)$ sind die Zusammenhangskomponenten von $\pi_0(X)$

– $\pi_0(X)$ ist total unzusammenhängend

haben Funktor

$\pi_0 : \mathbf{Top} \to \mathbf{Set}$

$\pi_0(X) := \{\text{Menge der Zusammenhangskomponenten}\}$

$[x]$ - Komponente von x

$\pi_0(f)([x]) := [f(x)]$

- wohldefiniert

– $[x] = [y] \Rightarrow x \in [y] \Rightarrow f(x) \in f([y]) \Rightarrow f([y]) \subseteq [f(x)] \Rightarrow [f(y)] = [f(x)]$

Lemma 2.82. *π_0 ist Homotopieinvariant.*

Proof.

$f_0, f_1 : X \to Y$

- $f_0 \overset{H}{\sim} f_1$
- x in X
- $H_x : [0,1] \to Y$, $H_x(t) = H(t,x)$
- $H_x([0,1])$ ist zusammenhängend
- $[f_0(x)] \in H_x([0,1]) \subseteq [f_0(x)]$
- $f_1(x) \in H_x([0,1]) \subseteq [f_0(x)]$
- also $\pi_0(f_1)([x]) = [f_1(x)] = [f_0(x)] = \pi_0(f_0)([x])$ □

X - top. space

Definition 2.83. *X is called contractible if the exists a point x in X such that $\{x\} \to X$ is a homotopy equivalence.*

- a contractible space is not empty

Wenn X kontrahierbar ist, dann ist $|\pi_0(X)| = 1$

- x in X
- $i : \{x\} \to X$ ist Homotopieäquivalenz
- $1 = |\pi_0(\{x\})| = |\pi_0(X)|$

Anwendung

S, T Mengen

- S_{disc} und T_{disc} sind genau dann homotopieäquivalent, wenn $|S| = |T|$ gilt.
- wenn $|S| = |T|$, dann existiert Bijektion $f : S \to T$, ist Isomorphism $f : S_{\mathrm{disc}} \to T_{\mathrm{disc}}$
- sei $f : S_{\mathrm{disc}} \to T_{\mathrm{disc}}$ eine Homotopieäquivalenz

– $\pi_0(f) : \pi_0(S_{\mathrm{disc}}) \to \pi_0(T_{\mathrm{disc}})$ ist Bijektion

– $\pi_0(S_{\mathrm{disc}}) \cong S$ und $\pi_0(T_{\mathrm{disc}}) \cong T$

– $|S| = |T|$

Anwendung

$[0, 1]$ und S^1 sind nicht isomorph.

- $|\pi_0([0, 1] \setminus \{1/2\})| = 2$
- $|\pi_0(S^1 \setminus \{u\})| = 1$ für jeden Punkt u in S^1, da $S^1 \setminus \{u\} \cong [0, 1]$

Beispiel:

definieren $f : \mathbb{Z} \to \pi_0(\mathtt{Map}(S^1, \mathbb{R}^2 \setminus \{0\}))$

- $\mathbb{R}^2 \cong \mathbb{C}$
- $f(n) := [S^1 \ni u \mapsto u^n \in \mathbb{R}^2]$
- werden später sehen: diese Abbildung ist eine Bijektion

– idea of proof using some analysis:

– identify $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{C} \setminus \{0\}$

– observe, that every class $[f]$ in $\texttt{Map}(S^1, \mathbb{C} \setminus \{0\})$ can be represented by a smooth map

– any two such smooth maps are smoothly homotopic

— actually $C^\infty(S^1, \mathbb{C} \setminus \{0\}) \to \texttt{Map}(S^1, \mathbb{C} \setminus \{0\})$ is a homotopy equivalence

— here $C^\infty(S^1, \mathbb{C} \setminus \{0\})$ with the topology of uniform convergence of all derivatives

– define map $d : \texttt{Map}(S^1, \mathbb{C} \setminus \{0\}) \to \mathbb{Z}$ by

– $[f] \mapsto \frac{1}{2\pi i} \int_{S^1} f^* \frac{dz}{z}$

— for $f(u) := u^n$

— $f^* \frac{dz}{z} = n du$

— $\frac{1}{2\pi i} \int_{S^1} f^* \frac{dz}{z} = \frac{n}{2\pi i} \int_{S^1} du = n$

- $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ ist keine Homotopieäquivalenz

- Konsequenz: wegen $S^1 \sim \mathbb{R}^2 \setminus \{0\}$ ist S^1 nicht kontrahierbar

- also $S^1 \not\sim [0, 1]$

3 Homologie

3.1 Paare

$\mathbf{C}$, $\mathbf{D}$ categories

- use notation $\mathbf{D}^{\mathbf{C}} := \mathbf{Fun}(\mathbf{C}, \mathbf{D})$ for the functor category

$\Delta^1 := (0 \to 1)$ - category

$\mathbf{Top}^{\Delta^1}$ - Kategorie der Morphismen in $\mathbf{Top}$

- explizite Beschreibung

– Objekte: Morphismen $X \stackrel{\phi}{\to} Y$ in $\mathbf{Top}$

– Morphismen: Paare $(f, g) : (X \stackrel{\phi}{\to} Y) \to (X' \stackrel{\phi'}{\to} Y')$ fitting into

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ Y & \xrightarrow{g} & Y' \end{array}$$

kommutiert

- Komposition in offensichtlicher Weise

Z - topologischer Raum

$Z \times - : \mathbf{Top}^{\Delta^1} \to \mathbf{Top}^{\Delta^1}$, $(X \xrightarrow{\phi} Y) \mapsto (Z \times X) \xrightarrow{\mathtt{id}_Z \times \phi} (Z \times Y)$

- use $Z = [0,1]$ in order to define notion of homotopy $\mathbf{hTop}^{\Delta^1}$

– $(f_0, g_0), (f_1, g_1) : (X \to Y) \to (X' \to Y')$

– $(f_0, g_0) \overset{(H,L)}{\sim} (f_1, g_1)$ falls

– $(H, L) : ([0,1] \times X, [0,1] \times Y) \to (X' \to Y')$

mit $(H, L)_{\{i\} \times (X \to Y)} = (f_i, g_i)$, $i = 0, 1$

- können homotopieinvariante Funktoren aus $\mathbf{Top}^{\Delta^1}$ betrachten

Definition 3.1. $\mathbf{Top}^2$ *ist die volle Unterkategorie von* $\mathbf{Top}^{\Delta^1}$ *aus den Einbettungen* $U \to X$ *von Teilräumen.*

- schreiben (X, U) statt $U \to X$

- call (X, U) a pair

- for morphism $(f, g) : (X, U) \to (Y, V)$: dann ist $g = f_{|U}$ redundant, brauchen nur Bedingung $f(U) \subseteq V$

- schreiben deshalb $f : (X, U) \to (Y, V)$ statt $(f_{|V}, f)$

want to show that $\mathbf{Top}^2$ is complete and cocomplete

brauchen allgemeine Tatsache:

$R : \mathbf{C} \to \mathbf{D}$ Funktor

Definition 3.2. *R heißt voll-treu, wenn für je zwei Objekte C, C' in $\mathbf{C}$ die induzierte Abbildung*

$$\mathtt{Hom}_{\mathbf{C}}(C, C') \to \mathtt{Hom}_{\mathbf{D}}(R(C), R(C'))$$

eine Bijektion ist.

Beispiele:

- $\mathbf{Top} \to \mathbf{Set}$ ist nicht voll-treu

- $\mathrm{incl} : \mathbf{Top}^2 \to \mathbf{Top}^{\Delta^1}$ ist voll-treu

- $\mathbf{Set} \to \mathbf{Top}$, $X \mapsto X_{\mathrm{disc}}$ ist voll-treu

Lemma 3.3. *Annahme:*

1. *R ist voll-treu.*
2. *R hat einen linksadjungierten L (haben also Adjunktion $L : \mathbf{D} \leftrightarrows \mathbf{C} : R$)*

Sei $C : \mathbf{I} \to \mathbf{C}$ ein Diagramm. Dann gilt:

1. *Wenn $\mathtt{colim}_{\mathbf{I}} R(\mathbf{C})$ in $\mathbf{D}$ existiert, dann existiert $\mathtt{colim}_{\mathbf{I}} \mathbf{C}$ und es gilt*

$$\underset{\mathbf{I}}{\mathtt{colim}}\, \mathbf{C} \cong L(\underset{\mathbf{I}}{\mathtt{colim}}\, R(\mathbf{C})) \ .$$

2. *Wenn $\mathtt{lim}_{\mathbf{I}} R(C)$ in $\mathbf{D}$ existiert im wesentlichen Bild von R enthalten ist, dann dann existiert $\mathtt{lim}_{\mathbf{I}} C$ in $\mathbf{C}$ und es gilt*

$$\underset{\mathbf{I}}{\mathtt{lim}}\, C \cong L(\underset{\mathbf{I}}{\mathtt{lim}}\, R(C)) \ .$$

Proof.

zu (1)

C' in $\mathbf{C}$ beliebig

$$\begin{aligned} \mathtt{Hom}_{\mathbf{C}}(L(\underset{\mathbf{I}}{\mathtt{colim}}\, R(C)), C') &\cong \mathtt{Hom}_{\mathbf{D}}(\underset{\mathbf{I}}{\mathtt{colim}}\, R(C)), R(C')) \\ &\cong \underset{\mathbf{I}^{\mathrm{op}}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{D}}(R(C), R(C')) \\ &\cong \underset{\mathbf{I}^{\mathrm{op}}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{C}}(C, C') \end{aligned}$$

conclusion: $\mathtt{colim}_{\mathbf{I}} C$ exists and is represented by $L(\mathtt{colim}_{\mathbf{I}} R(C))$

Spezialfall: $\mathbf{I} = *$

- Kounit ist Isomorphismus $L(R(C)) \cong C$

zu (2)

nach Annahme finde D in $\mathbf{C}$ und Iso $R(D) \cong \mathtt{lim}_{\mathbf{I}} R(C)$

C' in $\mathbf{C}$ beliebig:

$$\begin{aligned} \mathtt{Hom}_{\mathbf{C}}(C', D) &\cong \mathtt{Hom}_{\mathbf{D}}(R(C'), R(D)) \\ &\cong \mathtt{Hom}_{\mathbf{D}}(R(C'), \underset{\mathbf{I}}{\mathtt{lim}}\, R(C)) \\ &\cong \underset{\mathbf{I}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{D}}(R(C'), R(C)) \\ &\cong \underset{\mathbf{I}}{\mathtt{lim}}\, \mathtt{Hom}_{\mathbf{C}}(C', C) \end{aligned}$$

conclusion: $\mathtt{lim}_{\mathbf{I}} C$ exists and is represented by D

□

Proposition 3.4. *1. Es gibt eine Adjunktion*

$$L : \mathbf{Top}^{\Delta^1} \leftrightarrows \mathbf{Top}^2 : \mathrm{incl}$$

mit $L(\phi : X \to Y) := (Y, \phi(X))$

2. $\mathbf{Top}^2$ *ist vollständig und kovollständig.*

Proof.

(1)

$$\mathtt{Hom}_{\mathbf{Top}^2}((Y, \phi(X)), ((U, V)) \cong \mathtt{Hom}_{\mathbf{Top}^{\Delta^1}}((\phi : X \to Y), (V \to U)))$$

- $f \mapsto (f \circ \phi, f)$
- $(g, h) \mapsto h$

– universal property of image

(2)

cocompleteness

- benutzen Adjunktion

$$L : \mathbf{Top}^{\Delta^1} \leftrightarrows \mathbf{Top}^2 : \mathrm{incl}$$

und Lemma 3.3

- incl is vollstreu
- $\mathbf{Top}^{\Delta^1}$ is kovollständig (**Top** is kovollständig and colimits are taken pointwise)

completeness

- (Y, X) in $(\mathbf{Top}^2)^{\mathbf{I}}$

– betrachten das als Diagramm $(\phi : X \to Y)$ in $\mathbf{Top}^{\Delta^1}$

–

– $\mathtt{lim}_{\mathbf{I}} \phi : \mathtt{lim}_{\mathbf{I}} X \to \mathtt{lim}_{\mathbf{I}} Y$ in $\mathbf{Top}^{\Delta}$

– zeigen daß dieses Objekt in $\mathbf{Top}^2$ enthalten ist und Lemma 3.3 anwenden

- in der Tat ist $\mathtt{lim}_{\mathbf{I}} \phi$ die Einbettung eines Unterraumes (Lemma 2.45)

3.2 Axiome für eine Homologietheorie

R - Ring

- $\mathbf{Mod}(R)$ - Kategorie der (linken) R-Moduln

$\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}} := \mathbf{Fun}(\mathbb{Z}_{\mathrm{disc}}, \mathbf{Mod}(R))$ - $\mathbb{Z}$-graduierte R-Moduln

- explicit description

– Objekte: $(A_n)_{n\in\mathbb{Z}}$ - Familien von R-Moduln

– Morphismen: $(f_n)_{n\in\mathbb{Z}} : (A_n)_{n\in\mathbb{Z}} \to (A'_n)_{n\in\mathbb{Z}}$,

– $f_n : A_n \to A'_n$ in $\mathtt{Hom}_{\mathbf{Mod}(R)}(A_n, A'_n)$

fix m in $\mathbb{N}$

- get fully-faithful embedding $\mathbf{Mod}(R) \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$

- $M \mapsto M[n] := (M_i)_{i\in\mathbb{Z}}$ with $M_i = \begin{cases} M & i = -n \\ 0 & i \neq n \end{cases}$

Schiftfunktor: $T : \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}} \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$

- $T((A_n)_{n\in\mathbb{Z}}) := (A_{n-1})_{n\in\mathbb{Z}}$

- $T((f_n)_{n\in\mathbb{Z}}) = (f_{n-1})_{n\in\mathbb{N}}$

- T is an isomorphism

– for k in $\mathbb{Z}$ schreiben of auch $T^k(-) := (-)[-k]$

– $M[k]_n = M_{k+n}$

$\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ ist abelsche Kategorie

- haben Begriff von exakter Sequenz

- $(A_n)_{n\in\mathbb{N}} \to (A'_n)_{n\in\mathbb{N}} \to (A''_n)_{n\in\mathbb{N}}$ ist exact, falls $A_n \to A'_n \to A''_n$ für alle n in $\mathbb{N}$ exakt ist

□

Betrachten einen Funktor $H : \mathbf{Top}^2 \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$

- erhalten functor $\mathbf{Top} \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ by restriction via functor $X \mapsto (X, \emptyset)$

- Notation: $H(X) := H(X, \emptyset)$

formulieren Axiome:

Axiom 3.5 (Homotopieinvarianz). *H ist Homotopieinvariant, falls für je zwei homotope Morphismen* $f_0, f_1 : (X, Y) \to (X', Y')$ *in* $\mathbf{Top}^2$ *gilt* $H(f_0) = H(f_1)$

Lemma 3.6. *Die folgende Aussagen sind äquivalent:*

1. *H ist homotopieinvariant*
2. $H([0,1] \times X, [0,1] \times Y) \to H(X, Y)$ *ist für alle Paare* (X, Y) *ein Isomorphismus.*
3. *H faktorisiert über* $\mathbf{hTop}^2$.

Proof. Übungsaufgabe □

Examples:

(W, U) in $\mathbf{Top}^2$ represents homotopy invariant functor $(X, Y) \mapsto \mathtt{Hom}_{\mathbf{hTop}^2}((W, U), (X, Y))$

- for k in $\mathbb{Z}$ can define functor $\mathbf{Top}^2 \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ by "linearization"

– $(X, Y) \mapsto R[\mathtt{Hom}_{\mathbf{hTop}^2}((W, U), (X, Y))][k]$

Example:

- set $C_{X \setminus Y}(X, \mathbb{Z}) := \{f : X \to \mathbb{Z}_{\mathrm{disc}} \mid f_{|Y} = 0\}$
- $(X, Y) \mapsto \mathtt{Hom}_{\mathbf{Mod(Ab)}}(C_{X \setminus Y}(X, \mathbb{Z}), R)[k]$

betrachten Paar (X, Y) und Teilraum U von Y

- erhalten Paar $(X \setminus U, Y \setminus U)$
- Morphismus $(X \setminus U, Y \setminus U) \to (X, Y)$

Axiom 3.7 (Ausschneidung). *Für jedes Paar* (X, Y) *und Teilraum* U *von* X *mit* $\bar{U} \subseteq \mathrm{int}(Y)$ *ist die induzierte Abbildung* $H(X \setminus U, Y \setminus U) \to H(X, Y)$ *ein Isomorphismus.*

non-example:

- $(X, Y) \mapsto \mathtt{Hom}_{\mathbf{Top}^2}(W, X \setminus Y)$ (not a functor)

example:

- fix W in $\mathbf{Top}$
- $(X, Y) \to (R[\mathtt{Hom}_{\mathbf{Top}}(W, X)]/R[\mathtt{Hom}_{\mathbf{Top}}(W, Y)])\,[k]$ satisfies excision
- $(X, Y) \to \mathtt{Hom}_{\mathbf{Ab}}(C_{X \setminus Y}(X, \mathbb{Z}), R)[k]$ satisfies excision

$(X_i, Y_i)_{i\in I}$ Familie in $\mathbf{Top}^2$

- $c_i : (X_i, Y_i) \to \bigsqcup_{i\in I}(X_i, Y_i)$ kanonische Abbildung
- $H(c_i) : H(X_i, Y_i) \to H(\bigsqcup_{i\in I}(X_i, Y_i)$
- $(H(c_i))_{i\in I}$ induziert

$\bigoplus_{i\in I} H(X_i, Y_i) \to H(\bigsqcup_{i\in I}(X_i, Y_i))$

Definition 3.8 (Additivity). *H ist additiv, falls die Abbildung*

$$\bigoplus_{i\in I} H(X_i, Y_i) \to H(\bigsqcup_{i\in I}(X_i, Y_i))$$

für jedes (X, Y) in $\mathbf{Top}^2$ ein Isomorphismus ist.

example:

- fix W in **Top**
- $(X, Y) \to (R[\mathtt{Hom}_{\mathbf{Top}}(W, X)]/R[\mathtt{Hom}_{\mathbf{Top}}(W, Y)])\,[k]$ satisfies additivity if W is connected

non-example:

- $R = \mathbb{Q}$
- $(X, Y) \mapsto \mathtt{Hom}_{\mathbf{Ab}}(C_{X\setminus Y}(X, \mathbb{Z}), \mathbb{Q})[k]$ is not additive

– reason: $C_{\coprod_{i\in \mathbf{I}} X_i\setminus Y_i}(\coprod_{i\in\mathbf{I}} X_i, \mathbb{Z}) \cong \prod_{i\in\mathbf{I}} C_{X_i\setminus Y_i}(X_i, \mathbb{Z})$

– but in general $\bigoplus_{i\in\mathbf{I}} \mathtt{Hom}(A_i, \mathbb{Q}) \to \mathtt{Hom}(\prod_{i\in\mathbf{I}} A_i, \mathbb{Q})$ is not an isomorphism

— $\dim_{\mathbb{Q}} \bigoplus_{\mathbb{N}} \mathtt{Hom}(\mathbb{Q}, \mathbb{Q})$ is countable

— $\dim_{\mathbb{Q}} \mathtt{Hom}(\prod_{\mathbb{N}} \mathbb{Q}, \mathbb{Q})$ is uncountable

haben Funktoren $\mathbf{Top}^2 \to \mathbf{Top}$, $(X, Y) \mapsto X$, $(X, Y) \mapsto Y$

und Funktor $\mathbf{Top} \to \mathbf{Top}^2$, $X \mapsto (X, \emptyset)$

- erhalten Funktor $\mathbf{Top}^2 \to \mathbf{Top}^2$, $(X, Y) \mapsto (X, \emptyset)$ mit natürlicher Transformation $(X, \emptyset) \to (X, Y)$
- $e : \mathbf{Top}^2 \to \mathbf{Top}^2$, $(X, Y) \mapsto (Y, \emptyset)$ mit natürlicher Transformation $(Y, \emptyset) \to (X, \emptyset)$

betrachten natürliche Transformation

$$\partial : H \to T \circ H \circ e$$

- $\partial_n : H_n(X, Y) \to H_{n-1}(Y, \emptyset)$
- für (X, Y) in $\mathbf{Top}^2$ erhalten funktorielles Diagram

$$H(Y) \to H(X) \to H(X, Y) \xrightarrow{\partial} H(Y)[-1]$$

Axiom 3.9 (Exaktheitsaxiom)**.** *Das Paar (H, ∂) erfüllt das Exaktheitsaxiom, wenn*

$$H(Y) \to H(X) \to H(X,Y) \xrightarrow{\partial} H(Y)[-1]$$

für alle (X, Y) in $\mathbf{Top}^2$ exakt ist.

- construction of examples satisfying exactness is more complicated, see later

Definition 3.10. *Eine Homologietheorie (mit Werten in $\mathbf{Mod}(R)$) ist ein Paar (H, ∂) aus einem Funktor $H : \mathbf{Top}^2 \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ und einer Transformation $\partial : H \to T \circ H \circ e$, welches homotopieinvariant ist und das Ausschneidungs, das Exaktheitsaxiom und das Additivitätsaxiom erfüllt.*

Beispiel: Nullfunktor ist eine Homologietheorie

Definition 3.11. *$H(*, \emptyset)$ in $\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ heißt die Koeffizienten.*

Theorem 3.12. *Für jedes M in $\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ existiert eine Homologietheorie $(H(-; M), \partial)$ mit Werten in $\mathbf{Mod}(R)$ mit einem Isomorphismus $H(*; M) \cong M$.*

Proof. Beweis später durch explizite Konstruktion □

Nehmen im folgenden an, daß $H(-)$ eine Homologietheorie mit den Koeffizienten M ist

3.3 Mayer-Vietoris sequence

X in $\mathbf{Top}$

- (U, V) - open covering

$$\begin{array}{ccc} U \cap V & \xrightarrow{k} & U \\ \downarrow h & & \downarrow i \\ V & \xrightarrow{j} & X \end{array}$$

- (H, ∂) - homology theory

Lemma 3.13 (Mayer-Vietoris sequence)**.** *We have a long exact sequence*

$$H(U \cap V) \xrightarrow{k \oplus -h} H(U) \oplus H(V) \xrightarrow{i+j} H(X) \xrightarrow{\delta} H(U \cap V)[-1]$$

where δ

Proof.

have a map of pairs $(V, U \cap V) \to (X, U)$

- get map of long exact sequences (exactness axiom)

$$\begin{array}{ccccccc}
H(U \cap V) & \xrightarrow{h} & H(V) & \xrightarrow{s} & H(V, U \cap V) & \xrightarrow{\partial_{(V,U \cap V)}} & H(U \cap V)[-1] \\
\downarrow k & & \downarrow j & & \cong \downarrow (j,k) & & \downarrow k \\
H(U) & \xrightarrow{i} & H(X) & \xrightarrow{r} & H(X, U) & \xrightarrow{\partial_{(X,U)}} & H(U)[-1]
\end{array}$$

- $V = X \setminus (X \setminus V)$
- $X \setminus V = \overline{X \setminus V} \subseteq U$
- get isomorphism by excision
- discussion:

define $\delta : H(X) \overset{r}{\to} H(X, U) \overset{(j,k)^{-1}}{\to} H(V, U \cap V) \overset{\partial_{(V,U \cap V}}{\to} H(U \cap V)[-1]$

Verifications:

- complex
- exact

- complex

– at $H(U) \oplus H(V)$

— $(i + j)(h, -k) = ih - jk = 0$ by commutativity

– at $H(X)$

— $\delta(i + j) = \partial_{(V,U \cap V)}(j, k)^{-1} r(i + j) \overset{ri=0}{=} \partial_{(V,U \cap V)}(j, k)^{-1} rj \overset{s=(j,k)^{-1}rj}{=} \partial_{(V,U \cap V)} s = 0$

– at $H(U \cap V)$

— $(k \oplus -h)\delta = (k \oplus -h)\partial_{(V,U \cap V)}(j, k)^{-1} r = k\partial_{(V,U \cap V)}(j, k)^{-1} r \oplus 0 = \partial_{(X,U)} r \oplus 0 = 0$

- exactness:

– at $H(U) \oplus H(V)$

— $(i + j)(u, v) = 0$

— $0 = r(i + j)(u, v) = (j, k)s(v)$, hence $s(v) = 0$

— find w in $H(U \cap V)$ with $h(w) = v$

— $i(u + k(w)) = -j(v) + ik(w) = -j(v) + jh(w) = -j(v) + j(v) = 0$

— find x in $H(X, U)[1]$ such that $\partial_{(X,U)} x = u + k(w)$

— set $w' := -w + \partial_{(V,U\cap V)}(j, k)^{-1} x$

—- $-h(w') = h(w) - h\partial_{(V,U\cap V)}(j, k)^{-1}(x) = v$

—- $k(w') = -k(w) + k\partial_{(V,U\cap V)}(j, k)^{-1}(x) = -k(w) + \partial_{(X,U)} x = u$

– at $H(X)$

— x in $H(X)$, $\delta(x) = 0$

— $\partial_{(V,U\cap V)}(j, k)^{-1} r(x) = 0$

— find v in $H(V)$ with $s(v) = (j, k)^{-1} r(x)$

— $(j, k)^{-1} r(x - j(v)) = (j, k)^{-1} r(x) - s(v) = 0$

— $r(x - j(v)) = 0$

— find u in $H(U)$ with $i(u) = x - j(v)$

— $(i + j)(u, v) = x$

– at $H(U \cap V)$

— w in $H(U \cap V)$, $(k \oplus -h)(w) = 0$

— find z in $H(V, U \cap V)[1]$ with $\partial_{(V,U\cap V)}(z) = w$

— $\partial_{(X,U)}(j, k)(z) = k(w) = 0$

— find x in $H(X)[1]$ such that $r(x) = (j, k)(z)$

— then $\delta(x) = w$ □

3.4 Basic calculations

Basic assumption:

$(H, \partial) : \mathbf{Top} \to \mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$ - Homology theory

- M in $\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$

$H(*) \cong M$ - coefficients

Corollary 3.14. *The inclusions $0 \to D^n \to \mathbb{R}^n$ induces isomorphisms*

$$M \cong H(*) \cong H(D^n) \cong H(\mathbb{R}^n) \ .$$

Proof.

use homotopy invariance

inclusions $* \to D^n \to \mathbb{R}^n$ are homotopy equivalences □

Lemma 3.15. *If X is a set, then $H^*(X_{\text{disc}}) \cong \bigoplus_X M$.*

Proof.

- $X \cong \coprod_X *$
- apply wedge axiom

□

$*$ is final object in **Top**

X - a space

- have unique map $p : X \to *$
- get induced map $p_* : H(X) \to H(*)$

Definition 3.16. *We define the reduced homology functor $\tilde{H} : \mathbf{Top} \to \mathbf{Mod}(R)^{\mathbb{Z}-\text{gr}}$ to be the functor $X \mapsto \texttt{ker}(p_* : H(X) \to H(*))$.*

- needs justification

x - a point in X

$i : * \to X$ inclusion of x

Lemma 3.17. *We have an isomorphism $H(X) \cong M \oplus \tilde{H}(X)$, where M is identified with the image of i_*.*

Proof. $p \circ i = \texttt{id}$

- $p_* \circ i_* = \texttt{id}$
- projection in to image of i_* is $i_* \circ p_*$
- projection into $\texttt{ker}(p_*)$ is $1 - i_* \circ p_*$

– $H(X) \cong M \oplus \tilde{H}(X)$ □

consider pair (X, U)

assume $x \in U$

Lemma 3.18. *The long exact sequence of the pair (X, U) naturally induces a long exact sequence*

$$H(X,U)[1] \xrightarrow{\partial} \tilde{H}(U) \to \tilde{H}(X) \to H(X,U) \ .$$

Proof.

long exact sequence of pair

$$H(X,U)[1] \xrightarrow{\partial} M \oplus \tilde{H}(U) \xrightarrow{\mathtt{id}_M \oplus \tilde{H}(U \to X)} M \oplus \tilde{H}(X) \xrightarrow{\gamma} H(X,U) \ .$$

- conclude: $\gamma_{|M} = 0$

- ∂ takes values in $\tilde{H}(U)$

□

Lemma 3.19. *If $U \to X$ is a homotopy equivalence, then $H(X, U) = 0$.*

Proof.

long exact sequence

$H(U) \xrightarrow{\cong} H(X) \to H(X,U) \to H(U)[-1] \xrightarrow{\cong} H(X)[-1]$ □

$H(\mathbb{R}^n \setminus \{0\}, S^{n-1}) = 0$

Lemma 3.20.

1. *For every n in $\mathbb{N}$ with $n \geq 1$ we have an isomorphism*

$$H(D^n, S^{n-1}) \overset{\partial_{(D^n,S^{n-1})}}{\cong} \tilde{H}(S^{n-1})[-1] \ .$$

2. *For every n in $\mathbb{N}$ with $n \geq 1$ we have an isomorphism*

$$\tilde{H}(S^n) \cong H(D^n, S^{n-1}) \ .$$

Proof.

(1)

long exact sequence of (D^n, S^{n-1})

$\tilde{H}(S^{n-1}) \xrightarrow{\alpha} \tilde{H}(D^n) \xrightarrow{\beta} H(D^n, S^{n-1}) \xrightarrow{\partial} H(S^{n-1})[-1]$

- $0 = \tilde{H}(*) \cong \tilde{H}(D^n)$

- get $H(D^n, S^{n-1}) \overset{\partial}{\cong} \tilde{H}(S^{n-1})[-1]$

(2)

S^n_+ - closed upper hemisphere in S^n

- $S^n_+ \cong D^n \simeq *$

consider pair sequence for (S^n, S^n_+)

$$\tilde{H}(S^n_+) \xrightarrow{\gamma} \tilde{H}(S^n) \to H(S^n, S^n_+) \xrightarrow{\partial} \tilde{H}(S^n_+)[-1]$$

- $\tilde{H}(S^n_+) \cong \tilde{H}(*) = 0$

– $\tilde{H}(S^n) \cong H(S^n, S^n_+)$

– consider excision for subset $\{x\} \subseteq S^n_+$, x - north pole

– $(D^n, S^{n-1}) \cong (S^n_-, S^{n-1}) \simeq (S^n \setminus \{x\}, (S^n_+ \setminus \{x\})$

— $H(D^n, S^{n-1}) \cong H(S^n \setminus \{x\}, (S^n_+ \setminus \{x\}) \simeq H(S^n, S^n_+)$

- conclude finally

– $\tilde{H}(S^n) \cong H(D^n, S^{n-1})$ □

Lemma 3.21.

1. *For every n in $\mathbb{N}$ with $n \geq 0$ we have $H(S^n) \cong M \oplus M[-n]$.*
2. *For every n in $\mathbb{N}$ with $n \geq 1$ we have $H(D^n, S^{n-1}) \cong M[-n]$.*

Proof.

induction by n:

$n = 0$

- $S^0 \cong * \sqcup *$

- $H(S^0) \cong H(*) \oplus H(*) \cong M \oplus M \cong M \oplus M[0]$

- $\tilde{H}(S^0) \cong M[0]$

- $H(D^1, S^0) \cong M[-1]$ (by Lemma 3.20, 1.)

assumption:

- $\tilde{H}(S^{n-1}) \cong M[-(n-1)]$

- $H(D^n, S^{n-1}) \cong M[-n]$

step $(n-1 \to n)$

- use Lemma 3.20, 2. for: $\tilde{H}(S^n) \cong H(D^n, S^n) \cong M[-n]$

- use Lemma 3.20, 1. for: $H(D^{n+1}, S^n) \cong M[-(n+1)]$

finally:

$H(S^n) \cong M \oplus \tilde{H}(S^n) \cong M \oplus M[-n]$

□

The following applications depends on existence of a homology theory (H, ∂) with $H(*) \cong M$ for some $M \neq 0$ in $\mathbf{Mod}(R)$

Corollary 3.22. *If S^n and S^m are homotopy equivalent, then $n = m$.*

Proof.

take $R = \mathbb{Z}$, $M = \mathbb{Z}$

- note that $\mathbb{Z}[0] \oplus \mathbb{Z}[n] \cong \mathbb{Z}[0] \oplus \mathbb{Z}[m]$ iff $n = m$

□

U open in $\mathbb{R}^n$, u in U

Lemma 3.23. $H(U, U \setminus \{u\}) \cong M[-n]$

Proof.

find r in $(0, \infty)$ such that $B(u,r) \subseteq U$

- $A := U \setminus B(u,r)$ is closed $B(u,r) = U \setminus A$
- excision: $H(B(u,r), B(u,r) \setminus \{u\}) \cong H(U, U \setminus \{u\})$
- also $H(B(u,r), B(u,r) \setminus \{u\}) \cong H(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\}) \cong H(D^n, S^{n-1}) \cong M[-n]$

– since $(\mathbb{R}^n, \mathbb{R}^n \setminus \{u\}) \simeq (D^n, S^{n-1})$

□

Corollary 3.24 (Invarianz der Dimension)**.** *Assume that U is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^m$ and that there exists a homeomorphism between U and V. Then $n = m$.*

Proof.

$f : U \to V$ - homeomorphism

- $f : (U, U \setminus \{u\}) \to (V, V \setminus \{f(u)\})$ - isomorphism of pairs
- $M[-n] \cong H(U, U \setminus \{u\}) \cong H(V, V \setminus \{f(u)\}) \cong M[-m]$
- hence $n = m$

□

3.5 Application of Mayer-Vietoris

S - finite subset of $\mathbb{R}^n$

Lemma 3.25. $\tilde{H}(\mathbb{R}^n \setminus S) \cong \bigoplus_S M[-(n-1)]$.

Proof.

induction by $|S|$

- $|S| = 1$
- $\mathbb{R}^n \setminus S \simeq S^{n-1}$
- $\tilde{H}(\mathbb{R}^n \setminus S) \cong \tilde{H}(S^{n-1}) \cong M[-(n-1)]$

step:

assume: result for $|S| = k-1$

consider now $|S| = k$

- can assume $S \cap [-1,1] \times \mathbb{R}^{n-1} = \emptyset$
- (can be satisfied after moving the points by a homotopy)
- $|S \cap \mathbb{R}^n_{\pm}| < k$ (the half plane separates S non-trivially)
- decompose $\mathbb{R}^n \setminus S$ by $U := ((-\infty,1) \times \mathbb{R}^{n-1}) \setminus S$ and $V := ((-1,\infty) \times \mathbb{R}^{n-1}) \setminus S$

– $U \cap V \cong (-1,1) \times \mathbb{R}^{n-1} \simeq *$

– $U \simeq \mathbb{R}^n \setminus S'$

– $V \simeq \mathbb{R}^n \setminus S''$

- MV-sequence

$H(*) \to H(U) \oplus H(V) \to H(\mathbb{R}^n \setminus S) \xrightarrow{\delta} H(*)[-1]$

explicit:

$M \to M \oplus \bigoplus_{S'} M[-(n+1)] \oplus M \oplus \bigoplus_{S''} M[-(n+1) \to M \oplus \tilde{H}(\mathbb{R}^n \setminus S) \xrightarrow{\partial} M$

split of base points

$\bigoplus_S M[-(n+1) \cong \bigoplus_{S'} M[-(n+1)] \oplus \bigoplus_{S''} M[-(n+1) \cong \tilde{H}(\mathbb{R}^n \setminus S)$

□

Torus

$T^2 \cong S^1 \times S^1$

- write first factor as $[-1,1]/\sim$ with $\sim$ generated by $-1 \sim 1$
- get $T^2 \cong ([-1,1] \times S^1)/\sim$

open covering:

- $U = (-1,1) \times S^1)$
- $V = (([-1,1] \setminus \{0\}) \times S^1)/\sim$
- $U \cap V = ((-1,1) \setminus \{0\}) \times S^1 \cong (-1,0) \times S^1 \sqcup (0,1) \times S^1$

– $U \simeq S^1$

– $V \simeq S^1$

– $U \cap V \simeq S^1 \sqcup S^1$

MV sequence

$$H(U \cap V) \stackrel{\alpha}{\to} H(U) \oplus H(V) \to H(T^2) \stackrel{\delta}{\to}$$

$$H(S^1) \oplus H(S^1) \stackrel{\alpha}{\to} H(S^1) \oplus H(S^1) \to H(T^2) \stackrel{\delta}{\to}$$

$$\alpha(m,n) := (m+n, -m-n)$$

$$0 \to \mathrm{coker}(\alpha) \to H(T^2) \to \mathtt{ker}(\alpha)[-1] \to 0$$

$$\mathrm{coker}(\alpha) \cong H^1(S^1)\ , \quad (m,m') \mapsto m+m'$$

$$H^1(S^1) \cong \mathtt{ker}(\alpha)\ , \quad m \mapsto (m,-m)$$

explicitly:

$$0 \to M \oplus M[-1] \to H(T^2) \to M[-1] \oplus M[-2] \to 0$$

Exercise: show that this sequence splits:

more explicitly:

$M = \mathbb{Z}[0]$

- $H_0(T^2) \cong \mathbb{Z}$
- $H_1(T^2) \cong \mathbb{Z}^2$
- $H_2(T^2) \cong \mathbb{Z}$
- $H_i(T^2) = 0$ for $i \notin \{0, 1, 2\}$

glueing tori

consider T^2

- choose chart $\phi : U \cong \mathbb{R}^2$ at some point
- let $\Sigma_1^1 := T^2 \setminus \phi^{-1}(\mathrm{int}(D^2))$
- manifold with boundary $\partial \Sigma_1^1 \cong S^1$
- consider two copies and glue along boundary $\Sigma_2 := \Sigma_1^1 \sqcup_{S^1} \Sigma_1^1$
- Σ_2 - surface of genus 2
- want to calculate homology

Mayer Vietoris

$$\tilde{H}(S^1) \stackrel{(k,-h)}{\to} \tilde{H}(\Sigma_1^1) \oplus \tilde{H}(\Sigma_1^1) \to \tilde{H}(\Sigma_2) \stackrel{\delta}{\to} \tilde{H}(S^1)[-1]$$

- calculate $H(\Sigma_1^1)$

– represent T^2 as $([-1, 1] \times [-1, 1])/\sim$ with $(-1, u) \sim (1, u)$ and $(u, -1) \sim (u, 1)$ for all u in $[-1, 1]$

– take for D^2 small disc arround $(0, 0)$

– see: Σ_2^1 is homotopy equivalent to boundary $S^1 \sqcup_* S^1$ (look at picture)

– Mayer-Vietoris (exercise)

– $\tilde{H}(\Sigma_2^1) \cong \tilde{H}(S^1 \sqcup_* S^1) \cong M[-1] \oplus M[-1]$

– $i : S^1 \to \Sigma_2^1$ - inclusion of boundary of disc

– induces map $i' : S^1 \to S^1 \sqcup_* S^1$ (by composing with the homotopy equivalence)

– calculate $i_* : \tilde{H}(S^1) \to \tilde{H}(\Sigma_2^1)$

– equivalently $i'_* : \tilde{H}(S^1) \to \tilde{H}(S^1 \sqcup_* S^1)$

– claim $i'_* = 0$

– $p_0 : S^1 \sqcup_* S^1 \to S^1$ - identity on first copy, constant on second

– $p_0 \circ i' \sim \texttt{const}$ (see picture)

– similar for second copy

$(k, -h) = 0$

- MV sequence yields

- $0 \to M[-1]^{\oplus 4} \to \tilde{H}(\Sigma_2) \to M[-2] \to 0$

if $M = \mathbb{Z}[0]$

- $H_0(\Sigma_2) \cong \mathbb{Z}$

- $H_1(\Sigma_2) \cong \mathbb{Z}^4$

- $H_2(\Sigma_2) \cong \mathbb{Z}$

- $H_k(\Sigma_2) \cong 0$ for $k \notin \{0, 1, 2\}$

3.6 Mapping degree

take $R = \mathbb{Z}$, $\mathbf{Mod}(R) = \mathbf{Ab}$, $M = \mathbb{Z}[0]$

have iso of monoids $\mathtt{End}_{\mathbf{Mod}(\mathbb{Z})}(\mathbb{Z}) \cong \mathbb{Z}$ via $\phi \mapsto \phi(1)$

fix n in $\mathbb{N}$

- $H : \mathtt{End}_{\mathbf{Top}}(X, X) \to \mathtt{End}_{\mathbf{Ab}}(\tilde{H}_n(X))$ is map of monoids (by functoriality)

fix $n \geq 0$

Definition 3.26. *The mapping degree is the map of monoids* $\deg : \mathtt{End}_{\mathbf{Top}}(S^n) \to \mathbb{Z}$ *given by*

$$\mathtt{End}_{\mathbf{Top}}(S^n) \xrightarrow{\tilde{H}_n} \mathtt{End}_{\mathbf{Mod}(\mathbb{Z})}(\tilde{H}_n(S^n)) \cong \mathtt{End}_{\mathbf{Mod}(\mathbb{Z})}(\mathbb{Z}) \cong \mathbb{Z}$$

- note: $\deg(f)$ only depends on homotopy class of f

Example:

$n = 0$

$S^0 \cong *_0 \sqcup *_1$

- $\mathtt{End}_{\mathbf{Top}}(S^0) \cong \{\mathtt{id}, \sigma, p_0, p_1\}$

– $\sigma(*_i) := *_{1-i}$

– $p_i(*_j) := *_i$

- $\mathbb{Z} \cong \tilde{H}_0(S^0) \subseteq H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$
- included as $k \mapsto (k, -k)$

$\deg(\texttt{id}) = 1$

- $\sigma_*(k, -k) = (-k, k) = -(k, -k)$

conclusion: $\deg(\sigma) = -1$

- $p_0(k, -k) = (k - k, 0) = (0, 0)$
- $p_1(k, -k) = (0, -k + k) = (0, 0)$

conclusion: $\deg(p_i) = 0$

Example:

fix m

- $f : S^1 \to S^1$, $u \mapsto u^m$

Lemma 3.27. *We have* $\deg(f) = m$*.*

Proof.

$m = 0$ is clear

discuss case $m > 0$ in detail:

parametrize S^1 by $t \mapsto e^{2\pi i t}$

decompose $S^1 = A \cup B$,

– A is image of $(0, 1)$ and B is image of $(1/2, 3/2)$
- $f^{-1}(A) \cong A_1 \sqcup \cdots \sqcup A_m$, A_i is image of $((i - 1)/m, i/m)$

– $f_{|A_i} : A_i \to A$ homeomorphism
- $f^{-1}(B) \cong B_1 \sqcup \cdots \sqcup B_m$, B_i is image of $((i - 1/2)/m), (i + 1/2)/m)$

– $f_{|B_i} : B_i \to B$ homeomorphism

– $A \cap B = U \cup V$, U and V disjoint intervals

– $f^{-1}(A \cap B) = f^{-1}(U) \cup f^{-1}(V)$ (again unions of m intervals)
- Mayer-Vietoris sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(f^{-1}(A\cap B)) & \longrightarrow & H_0(f^{-1}(A)) \oplus H_0(f^{-1}(B)) \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1(S^1) & \longrightarrow & H_0(A\cap B) & \longrightarrow & H_0(A) \oplus H_0(B)
\end{array}$$

explicitly

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\tilde{\alpha}} & \bigoplus_{i=1}^m \mathbb{Z} \oplus \bigoplus_{i=1}^m \mathbb{Z} & \longrightarrow & \bigoplus_{i=1}^m \mathbb{Z} \oplus \bigoplus_{i=1}^m \mathbb{Z} \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle c} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}
\end{array}$$

description of some maps

- $\tilde{\alpha} : 1 \mapsto (1 \oplus \cdots \oplus 1) \oplus (-1 \oplus \cdots \oplus -1)$
- $c : (u_1 \oplus \cdots \oplus u_m) \oplus (v_1 \oplus \cdots \oplus v_m) \mapsto (u_1 + \cdots + u_m) \oplus (v_1 + \cdots + v_m)$
- $\alpha : 1 \mapsto 1 \oplus -1$

– $c(\tilde{\alpha}(1)) = \alpha(m)$ implies the assertion

$\square$

$\mathbf{Top}_{*/}$ - pointed topological spaces

- adjunction $(-)_+ : \mathbf{Top} \leftrightarrows \mathbf{Top}_* : forget$
- $(-)_+ : X \mapsto X_+ := \sqcup *$
- $H(X) \cong \tilde{H}(X_+) \cong H(X_+, *)$

Definition 3.28.

1. *We define the cone functor* $C : \mathbf{Top} \to \mathbf{Top}_{*/}$ *such that it sends a space* X *to the push-out*

$$\begin{array}{ccc}
\{-1\} \times X & \longrightarrow & [-1,0] \times X \\
\downarrow & & \downarrow \\
* & \longrightarrow & C(X)
\end{array} .$$

2. *We define the reduced cone functor* $\tilde{C} : \mathbf{Top}_* \to \mathbf{Top}_{/*}$ *such that it sends a space* X *to the push-out*

$$\begin{array}{ccc}
(\{-1\} \times X) \cup ([-1,0] \times *) & \longrightarrow & [-1,0] \times X \\
\downarrow & & \downarrow \\
* & \longrightarrow & C(X)
\end{array} .$$

3. *We define the suspension functor* $\Sigma : \mathbf{Top} \to \mathbf{Top}$ *such that it sends a space* X *to the space defined by the push-out*

$$\begin{array}{ccc} \{-1,1\} \times X & \longrightarrow & [-1,1] \times X \\ \downarrow & & \downarrow \\ \{-1,1\} & \longrightarrow & \Sigma X \end{array} \quad .$$

4. *We further define the reduced suspension* $\hat{\Sigma} : \mathbf{Top} \to \mathbf{Top}$ *such that it sends a space* X *to the pushout*

$$\begin{array}{ccc} \{-1,1\} \times X & \longrightarrow & [-1,1] \times X \\ \downarrow & & \downarrow \\ * & \longrightarrow & \hat{\Sigma} X \end{array} \quad .$$

5. *We define the reduced suspension functor* $\tilde{\Sigma} : \mathbf{Top}_* \to \mathbf{Top}_*$ *such that it sends a pointed space* X *to the push-out*

$$\begin{array}{ccc} (\{-1,1\} \times X) \cup ([1,1] \times *) & \longrightarrow & [-1,1] \times X \\ \downarrow & & \downarrow \\ * & \longrightarrow & \tilde{\Sigma} X \end{array} \quad .$$

note:

- $[-1, x]$ is called cone tip

– can consider $C : \mathbf{Top} \to \mathbf{Top}_{*/}$

- $C(X)$ is contractible

- have embedding $X \to C(X)$, $x \mapsto [0, x]$ (as cone base)

- $C(X) \cong \tilde{C}(X_+)$

- $\Sigma X \cong ([-1,1] \times X)/\sim$ with $\sim$ generated by $(\pm 1, x) \sim (\pm 1, x')$ for all x, x' in X

- $\Sigma X \cong C(X) \cup_X C(X)$ (glueing along the cone bases)

- have embedding $X \to \Sigma X$

$\hat{\Sigma} X := \Sigma X/\sim$ with $\sim$ is generated by $[(-1, x)] \sim [(1, x)]$ for all x in X

have embedding $X \to \tilde{C}(X)$, $x \mapsto (0, x)$

- $\tilde{\Sigma} X \cong \tilde{C}(X) \cup_X \tilde{C}(X)$

- $\hat{\Sigma} X \cong \tilde{\Sigma}(X_+)$

Lemma 3.29.

1. $\tilde{H}(\Sigma X) \cong \tilde{H}(X)[-1]$
2. $\tilde{H}(\hat{\Sigma} X) \cong H(X)[-1]$
3. $\tilde{H}(\tilde{\Sigma} X) \cong \tilde{H}(X)[-1]$ *if* $(X, *)$ *is well-pointed, see later*

Proof. cover ΣX by $U := \Sigma X \setminus \{[-1, x]\}$ and $V := \Sigma X \setminus \{[1, x]\}$

- U and V are contractible
- $U \cap V \simeq X$

MV-sequence

$$H(X) \to M \oplus M \to H(\Sigma X) \to H(X)[-1] \to$$

after reduction

$$\tilde{H}(\Sigma X) \stackrel{\delta}{\cong} \tilde{H}(X)[-1]$$

cover $\hat{X}$ by

- U - the image of $(-1, 1) \times X \simeq X$
- V - the image of $\hat{\Sigma} X \setminus \{0\} \times X \simeq *$
– then $U \cap V \cong ((-1, 0) \cup (0, 1)) \times X \simeq X \sqcup X$

MV-sequence

$$\cdots \to H(\hat{\Sigma} X)[1] \stackrel{\delta}{\to} H(X) \oplus H(X) \stackrel{\alpha:(a,b)\mapsto(a-b,p_*(a-b))}{\to} H(X) \oplus M \to H(\hat{\Sigma} X) \stackrel{\delta}{\to} \dots$$

– $\texttt{ker}(\alpha) \cong H(X)$, $a \mapsto (a, a)$
– $\mathrm{coker}(\alpha) \cong M$, $(a, m) \mapsto m$
– $0 \to M \to H(\hat{\Sigma} X) \stackrel{\delta}{\to} H(X)[-1] \to 0$
– summand M splits of (contribution of base point)
- conclude $\tilde{H}(\hat{\Sigma} X) \cong H(X)[-1]$

have projection map $q : \hat{\Sigma} \to \tilde{\Sigma} X$

- let $H : [0, 1] \times U \to U$ deformation retraction of neighbourhood U of $*$
- define map $j : \tilde{\Sigma} X \to \hat{\Sigma}$ by

$$j(u,x) := \begin{cases} (u,x) & x \notin U \\ H(|u|,x) & x \in U \end{cases}$$

cover $\tilde{\Sigma}X$ by

- - then U and V are contractible

- $U \cong (-1,1) \times X \simeq X$

- $U \cap V \cong (-1,1) \times X/ \sim \simeq X$ with $(u,*) \sim (0,=)$ for all u in $(-1,1)$

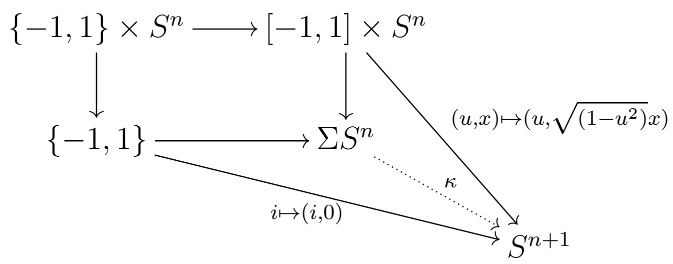
Mayer-Vietoris

- $H(\tilde{\Sigma}X,*) \xrightarrow{\delta} H(X,*)[-1]$

$\tilde{H}(\hat{\Sigma}X) \cong H(\hat{\Sigma}X,*) \cong H(\tilde{\Sigma}(X_+),*) \cong H(X_+,*)[-1] \cong H(X)[-1]$ □

Lemma 3.30. *For every n in $\mathbb{N}$ we have a homeomorphism $\Sigma S^n \cong S^{n+1}$.*

Proof. pushout



observe:

- dotted arrow is bijection

- target is Hausdorff

- domain is quasi-compact

- hence dotted arrow is homeomorphism.

□

$f : S^n \to S^n$

consider $\kappa \circ \Sigma(f) \circ \kappa^{-1} =: \sigma(f) : S^{n+1} \to S^{n+1}$

Lemma 3.31. *We have the equality* $\deg(\sigma(f)) = \deg(f)$.

Proof.

covering of ΣS^n by U and V as above is compatible with $\Sigma(f)$

$$\begin{array}{ccccc} H_{n+1}(S^{n+1}) & \xleftarrow[\kappa_*]{\cong} & H(\Sigma S^n) & \xrightarrow[\cong]{\partial} & H_n(S^n) \\ \downarrow{\scriptstyle \sigma(f)_*} & & \downarrow{\scriptstyle \Sigma(f)} & & \downarrow{\scriptstyle f_*} \\ H_{n+1}(S^{n+1}) & \xleftarrow[\kappa_*]{\cong} & H(\Sigma S^n) & \xrightarrow[\cong]{\partial} & H_n(S^n) \end{array}$$

□

n in $\mathbb{N}$, $n \geq 1$

Corollary 3.32. $\deg : \mathtt{End}_{\mathbf{Top}}(S^n) \to \mathbb{Z}$ *is surjective.*

write $\Sigma(f)$ instead of $\sigma(f)$ from now one

Example:

$f := \mathtt{diag}(1, \ldots, 1, -1) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

- induces $S^n \to S^n$ by restriction

- observe $f = \Sigma^{n+1}(\sigma)$ for σ in $\mathtt{End}(S^0)$

conclude: $\deg(f) = -1$

$A \in GL(n+1, \mathbb{R})$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$

Corollary 3.33. *A acts on $H_{n+1}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\})$ by multiplication by $\mathtt{sign}(\det(A))$.*

Proof. $GL(n+1, \mathbb{R})$ has two components distinguished by sign of $\det(A)$

- either $A \simeq \mathtt{id}$ (if $\det(A) > 0$) or $\det(A) \simeq \mathtt{diag}(1, \ldots, 1, -1)$ (if $\det(A) < 0$)

$H_{n+1}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}) \cong H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong H_n(S^n)$

- if $\det(A) > 0$, then A acts by $\deg(\mathtt{id}) = 1$

- if $\det(A) < 0$, then A acts by $\deg(\mathtt{diag}(1, \ldots, 1, -1)) = -1$ □

3.7 Fundamental classes

(H, ∂) - homology theory

- $H(*) =: M$

X - topological manifold

x in X

- $n := \dim_x(X)$

Lemma 3.34. *We have an isomorphism* $H(X, X \setminus \{x\}) \cong M[-n]$

Proof.

choose chart $f : U \xrightarrow{\cong} \mathbb{R}^n$ with $f(x) = 0$

excision

- cut out $X \setminus f^{-1}(D^n)$

$H(X, X \setminus \{x\}) \cong H(f^{-1}(D^n), f^{-1}(D^n \setminus \{0\})) \xrightarrow{f_*} H(D^n, D^n \setminus \{0\}) \cong H(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong M[-n]$ □

note: the isomorphism depends on the choice of chart f

consider (H, ∂) with $H(*) \cong \mathbb{Z}[0]$

- recall: $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$

X - topological manifold

- $n := \dim(X)$ (pure dimension)
- for x in X set $r_x : H_n(X) \to H_n(X, X \setminus \{x\})$

Definition 3.35. *A fundamental class of* X *is a class* $[X] \in H_n(X)$ *such that* $r_x([X])$ *is a generator for all* x *in* X*.*

Definition 3.36. *A homologically oriented manifold is a pair* $(X, [X])$ *of manifold* X *and a fundamental class* $[X]$*.*

Examples:

S^n

recall: $H_n(S^n) \cong \mathbb{Z}$

- consider pair sequence for $(S^n, S^n \setminus \{x\})$

$H_n(S^n) \xrightarrow{r_x} H_n(S^n, S^n \setminus \{x\}) \xrightarrow{\partial} H_{n-1}(S^n \setminus \{x\}) \xrightarrow{\beta} H_{n-1}(S^n) = 0$

– claim: $\partial = 0$

— two cases:

—- if $n > 1$: $H_{n-1}(S^n \setminus \{x\}) \cong 0$

— $n = 1$: β is injective

- conclude r_x is surjective

- choose for $[S^n]$ a generator of $H_n(S^n) \cong \mathbb{Z}$

– then $r_x([S^n])$ is a generator (x arbitrary)

– hence $[S^n]$ is a fundamental class

T^2

have calculated: $H_2(T^2) \cong \mathbb{Z}$

- choose generator $[T^2]$

- x in T^2

- consider pair sequence

$$H_n(T^2) \xrightarrow{r_x} H_n(T^2, T^2 \setminus \{x\}) \xrightarrow{\partial} H_1(T^2 \setminus \{x\}) \xrightarrow{i} H_1(T^2)$$

- $T^2 \setminus \{x\} \simeq S^1 \sqcup_* S^1$

- i isomorphism

- $\partial = 0$

- r_x is surjective

- hence $[T^2]$ is fundamental class

Lemma 3.37. *An oriented smooth manifold has a preferred fundamental class.*

Proof. Later □

assume: X is connected

- $\mathtt{ker}(r_x)$ is independent of x by homotopy invariance

- if $[X]$ is a fundamental class, then $H_n(X) \cong \mathbb{Z}[X] \oplus \mathtt{ker}(r_x)$

$(X, [X])$, $(Y, [Y])$ - oriented topological manifolds of dimension n

- $f : X \to Y$

Definition 3.38. *Assume that Y is connected. The degree of f is the number* $\deg(f) \in \mathbb{Z}$ *uniquely defined by* $f_*([X]) = \deg(f)[Y] + a$ *with* $a \in \mathtt{ker}(r_y)$.

- $\deg(f)$ is the image of $[X]$ under

$$H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{r_y} H_n(Y, Y \setminus \{y\}) \overset{r_y([Y])\mapsto 1}{\cong} \mathbb{Z}$$

this generalizes the degree for maps $S^n \to S^n$

X - topological Hausdorff space

S discrete closed subset

- for s in S have map $i_s : (X, X \setminus S) \to (X, X \setminus \{s\})$

Lemma 3.39. *We have an isomorphism* $\oplus_{s\in S} i_{s,*} : H(X, X \setminus S) \to \bigoplus_{s\in S} H(X, X \setminus \{s\})$.

Proof.

X is Hausdorff

- we can find pairwise disjoint family $(U_s)_{s\in S}$ of open subsets such that $S \cap U_s = \{s\}$ for all s in S

- set:

– $U := \bigcup_{s\in S} U_s$

– $V := X \setminus S$

- then (U, V) is an open covering of X

relative MV-Sequence

$$H(U\cap V, (U\cap V)\setminus S) \to H(U, U\setminus S)\oplus H(V, V\setminus S) \to H(X, X\setminus S) \to H(U\cap V, (U\cap V)\setminus S)[-1]$$

- $(U \cap V) \setminus S = U \cap V$

- $V \setminus S = V$

- get simplification

$$H(U, U \setminus S) \cong H(X, X \setminus S)$$

- wedge axiom and $S \cap U_s = \{s\}$:

$$H(U, U \setminus S) \cong \bigoplus_{s \in S} H(U_s, U_s \setminus \{s\})$$

excision (cut out $X \setminus U_s$)

$$H(U_s, U_s \setminus \{s\}) \cong H(X, X \setminus \{s\})$$

□

$(X, [X])$ and $(Y, [Y])$ homologically oriented topological manifolds

$f : X \to Y$

local degree

x in X

$y := f(x)$ in Y

- assume: $f^{-1}(y)$ is discrete (automatically closed!)

Definition 3.40. *We define the local degree* $\deg_x(f)$ *by the commuting diagram*

$$\begin{array}{ccccccc}
H_n(X, X \setminus \{x\}) & \longrightarrow & \bigoplus_{x' \in f^{-1}(y)} H(X, X \setminus \{x'\}) & \xrightarrow[\cong]{exc.} & H(X, X \setminus f^{-1}(\{y\})) & \xrightarrow{f_*} & H(Y, Y \setminus \{y\}) \\
\cong \downarrow r_x([X]) \mapsto 1 & & & & & & \cong \downarrow r_y([Y]) \mapsto 1 \\
\mathbb{Z} & & & \xrightarrow{\deg_x(f)} & & & \mathbb{Z}
\end{array}$$

fix y in Y

Proposition 3.41. *Assume that* Y *is connected. If* $f^{-1}(\{y\})$ *is discrete, then*

$$\deg(f) \cong \sum_{x \in f^{-1}(\{y)\})} \deg_x(f) \ .$$

Proof.

$$\begin{array}{ccccccccc}
H_n(X) & \longrightarrow & H_n(X, X \setminus f^{-1}(\{y\})) & \xrightarrow[excision]{\cong} & \bigoplus_{x \in f^{-1}(\{y\})} H(X, X \setminus \{x\}) & \xrightarrow[(\deg_x(f))_{x \in f^{-1}(y)}]{\cong} & \bigoplus_{x \in f^{-1}(\{y\})} \mathbb{Z} \\
\downarrow f_* & & \downarrow f_* & & & & \downarrow + \\
H_n(Y) & \longrightarrow & H_n(Y, Y \setminus \{y\}) & & \xrightarrow[\cong]{r_y([Y]) \mapsto 1} & & \mathbb{Z}
\end{array}$$

right pentagon commutes by definition of local degree □

Corollary 3.42. *If* $\deg(f) \neq 0$*, then* f *is surjective.*

application:

$f : \mathbb{R}^n \to \mathbb{R}^n$ proper map

- f has continuous extension $\tilde{f} : S^n \to S^n$

Corollary 3.43. *If* $\deg(\tilde{f}) \neq 0$*, then for every* y *in* $\mathbb{R}^n$ *there exists* x *in* $\mathbb{R}^n$ *such that* $f(x) = y$*.*

note: the degree of $\tilde{f}$ can be determined by looking at the preimage of one point x_0

- get a conclusion about the preimages of all points

special case (Zwischenwertsatz):

- $f : [-1, 1] \to [-1, 1]$, $f(-1) = -1$ and $f(1) = 1$
- extend f to map $\mathbb{R} \to \mathbb{R}$ by $f(t) := t$ for $t \notin [-1, 1]$
- $\deg(\tilde{f}) = 1$ (look e.g. at 2: $f^{-1}(2) = 2$, $\deg_2(f) = 1$, since local homeo)
- hence for every u in $[-1, 1]$ there exists v in $[-1, 1]$ such that $f(v) = u$ (Zwischenwertsatz)

how to calculate local degree

- choose neighbourhood U of x such that $U \cap f^{-1}(\{y\}) = \{x\}$
- $H_n(X) \to H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}) \to H_n(Y, Y \setminus \{f(x)\}) \cong \mathbb{Z}$
- sends $[X]$ to $\deg_x(f)$
- X goes to generator of $H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$
- $\deg_x(f)$ only depends on $f_{|U}$

consider smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$

- assume $f(0) = 0$

Lemma 3.44. $f \sim df(0)$

Proof.

construct homotopy H_t

- for $t \in (0, 1]$ define $H(t, x) := t^{-1} f(tx)$
- extends continuously to $t = 0$ with $H(0, x) := df(0)(x)$

□

Corollary 3.45. f_* *acts on* $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ *by multiplication by* $\texttt{sign}(\det(df(0)))$

$(X, [X])$ and $(Y, [Y])$ homologically oriented topological manifolds

$f : X \to Y$

assume

- Y is smooth near y
- X is smooth near x in $f^{-1}(y)$
- f is smooth near x
- $df(x)$ is an isomorphism

- can choose charts:
  – U at x sending x to 0
  – V at y sending y to 0
  – $f_{|U} : U \to V$ is diffeomorphism represented by $f_{V,U} : \mathbb{R}^n \to \mathbb{R}^n$
- in this chart: $[X]$ goes to s_x and $[Y]$ goes to s_y
  – $s_x, s_y \in \{1, -1\}$
- $f_{V,U}$ - representative in charts

Lemma 3.46. $\deg_x(f) = \mathtt{sign}(\det(df_{V,U}(x)))s_x s_y$

Proof.

- $f_{V,U}$ is homotopic to linear map $df_{V,U}(0)$
- $\deg_x(f) = \deg(df_{V,U}(0))$
- justification

$$\begin{array}{ccccc}
H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & & \xrightarrow{\quad f_{V,U} \quad} & & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\} \\
\downarrow \cong & & & & \downarrow \cong \\
H_n(U, U \setminus \{x\}) & & \xrightarrow{\quad f_{|U} \quad} & & H_n(V, V \setminus \{y\}) \\
\downarrow \cong & & & & \downarrow \cong \\
H_n(X, X \setminus \{x\}) & \xrightarrow{exc} & H(X, X \setminus f^{-1}(y)) & \longrightarrow & H_n(Y, Y \setminus \{x\}) \\
\downarrow & & & & \downarrow \\
\mathbb{Z} & & \longrightarrow & & \mathbb{Z}
\end{array}$$

□

$(X, [X])$ and $(Y, [Y])$ smooth oriented manifolds

$f : X \to Y$ smooth map

y in Y

Definition 3.47. *y is called a regular value if $df(x)$ is surjective for all x in $f^{-1}(y)$.*

- note that regular values of f are dense (full measure) in Y

Corollary 3.48. *Assume that Y is connected and that $\dim(X) = \dim(Y)$. If y is a regular value of f, then $|f^{-1}(y)| \geq \deg(f)$.*

Proof.

absolute values of local degrees bounded by 1

□

$f(u) := u^m : S^1 \to S^1$

- chart $t \mapsto e^{2\pi i t}$
- in chart $f(t) = mt$
- $df(t) = m$
- $\deg_u(f) = \begin{cases} 0 & m = 0 \\ 1 & m \geq 1 \\ -1 & m \leq -1 \end{cases}$
- -1 has $|m|$ preimages
- $\deg(f) = |m|\mathtt{sign}(m) = m$

orientation classes of submanifolds

- $i : X \to \mathbb{R}^{n+k}$ - embedding of compact codimension k submanifold
- $N \to X$ - normal bundle $N := (X \times \mathbb{R}^{n+k})/\mathtt{im}(di)$

– if X is defined by global function $g : \mathbb{R}^{n+k} \to \mathbb{R}^k$, then N is trivialized by $\mathrm{grad}(g)$

- get extension of embedding

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{R}^{n+k} \\ \downarrow{\scriptstyle 0} & \nearrow{\scriptstyle \tilde{i}} & \\ N & & \end{array}$$

- uses geometry: identify $N \cong TX^\perp$, define map $N \to \mathbb{R}^n$ by $N_x \ni n \mapsto x + n \in \mathbb{R}^{n+k}$, calculate differential and show that it is invertible at zero section, induces embedding on some open disc bundle, rescale in order to identify disc bundle with N

- $D(N)$ - unit disc bundle, manifold with boundary $S(N)$

- $\tilde{i} : D(N) \to \mathbb{R}^{n+k}$ - embedding as codimension 0-manifold with boundary

Definition 3.49. *$X^N := D(N)/S(N)$ is called the Thom space of $N \to X$.*

clutching map $c : S^{n+k} \to X^N$

- view $\mathbb{R}^{n+k}$ as subspace of S^{n+k}

- $c(x) := \begin{cases} \tilde{i}^{-1}(x) & x \in \tilde{i}(D(N) \setminus S(N)) \\ * & else \end{cases}$

induces $c_* : H(S^{n+k}) \to H(X^N)$

Proposition 3.50 (Thom Isomorphism Theorem)**.** *If X is oriented, then we have an isomorphism*

$$\tilde{H}(X^N) \cong H(X)[-k] \ .$$

Proof.

general case: later

special case: if N is trivial:

- observe:

- $D(N) \cong [-1,1]^k \times X$

- $X^N \cong \tilde{\Sigma}^k(X_+)$

– use $\tilde{H}(\tilde{\Sigma}Y) \cong \tilde{H}(Y)[-1]$ for well-pointed space Y (proof and definition later)

– all space appearing below are well-pointed

- $\tilde{H}(X^N) \cong \tilde{H}(\tilde{\Sigma}^k(X_+)) \cong \tilde{H}(\tilde{\Sigma}^{k-1}(X_+))[-1] \cong \ldots \cong \tilde{H}(X_+)[-k] \cong H(X)[-k]$ □

Proposition 3.51. *If X is oriented, then the image of 1 under*

$$\mathbb{Z} \cong \tilde{H}_{n+k}(S^{n+k}) \stackrel{c_*}{\to} \tilde{H}_{n+k}(X^N) \stackrel{Thom\, iso}{\to} H_n(X)$$

is a fundamental class of X.

Proof. later, will do the case $k = 1$ below □

X - compact codimension one sumanifold

- assume that normal bundle is trivial
- $D(N) \cong X \times [-1, 1]$
- get embedding $X \times [-1, 1] \to \mathbb{R}^{n+1}$

– make picture

- write $T := D(N)/S(N) \cong \tilde{\Sigma}(X_+) \cong \hat{\Sigma}X$
- have isomorphism $H_{n+1}(T) \cong H_n(X)$

– special case of Thom isomorphism

- $H_{n+1}(S^{n+1}) \xrightarrow{c_*} H_{n+1}(T) \to H_n(X)$
- set $[X] :=$ image of 1 in $H_n(X)$

Lemma 3.52. $[X]$ *is a fundamental class.*

Proof.

coordinates (x^0, x') of $\mathbb{R} \times \mathbb{R}^n \cong \mathbb{R}^{n+1}$

x in X

- can assume that neighbourhood W of x is contained in $\mathbb{R}^n = x^0 = 0$, $x = (0, 0')$

cover $\hat{\Sigma}X$

- $U = (-1, 1) \times X$
- $V = \hat{\Sigma}X \setminus X$
- $U \cap V \simeq X \sqcup X$
- $\mathtt{pr}_1 H(U \cap V) \to H(X)$ - projection onto first component

get upper two lines of

$$
\begin{array}{ccccccc}
H_{n+1}(S^{n+1}) & \xrightarrow{c_*} & H_{n+1}(T) & \xrightarrow{\delta} & H_n(U \cap V) & \xrightarrow{\mathtt{pr}_1} & H_n(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow r_x \\
H_{n+1}(S^{n+1}, S^{n+1} \setminus \Sigma\{x\}) & \longrightarrow & H_{n+1}(T, T \setminus \hat{\Sigma}\{x\}) & \longrightarrow & H_n(U \cap V, (U \cap V) \setminus \hat{\Sigma}\{x\}) & \longrightarrow & H_n(X, X \setminus \{x\}) \\
\| & & \cong \uparrow exc & & \cong \uparrow exc & & \cong \uparrow exc \\
 & & H_{n+1}(\hat{\Sigma}W, \hat{\Sigma}W \setminus \hat{\Sigma}\{x\}) & \longrightarrow & H_n(W \sqcup W, (W \setminus \{x\}) \sqcup (W \setminus \{x\})) & \longrightarrow & H_n(W, W \setminus \{x\}) \\
\| & & \cong \downarrow exc & & \cong \downarrow exc & & \cong \downarrow exc \\
H_{n+1}(S^{n+1}, S^{n+1} \setminus \Sigma\{x\}) & \xrightarrow{\cong} & H_{n+1}(\hat{T}, \hat{T} \setminus \hat{\Sigma}\{x\}) & \longrightarrow & H_n(\hat{U} \cap \hat{V}, (\hat{U} \cap \hat{V}) \setminus \hat{\Sigma}\{x\}) & \longrightarrow & H_n(S^n, S^n \setminus \{x\}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_{n+1}(S^{n+1}) & \longrightarrow & H_{n+1}(\hat{T}) & \longrightarrow & H_n(\hat{U} \cap \hat{V}) & \longrightarrow & H_n(S^n)
\end{array}
$$

$$H_{n+1}(S^{n+1}) \xrightarrow{\cong} H_n(S^n)$$

$\hat{T} := S^{n+1}/((-1, 0') \sim (1, 0'))$

– $\hat{c}_*$ is projection to quotient

- $\hat{U} := S^{n+1} \setminus \{(-1, 0')(1, 0')\}$

- $\hat{V} := \hat{T} \setminus S^n$

dotted arrow is a factorization of the clutching map

use MV for decomposition of S^{n+1} in order to coclude that lower horizontal map is iso

- conclude that 1 in $H_{n+1}(S^{n+1})$ goes to generator

□

Gauss map

- assume X is globally defined by $g : \mathbb{R}^{n+1} \to \mathbb{R}$

- g determines unit normal vector field

– $\nu : \text{grad}^0(g) := \frac{\text{grad}(g)}{\|\text{grad}(g)\|} : X \to S^n$

Definition 3.53. *The map $\nu : X \to S^n$ is called the Gauss map of X.*

Question: Calculate the degree of the Gauss map $\nu : (X, [X]) \to (S^n, [S^n])$

case $n = 2$:

- vertical embedding Σ_k

- look at coordinate function $f := (x^3)_{|\Sigma_k}$ restricted to Σ_k

- consider local degree at preimage of $(0,0,1)$

- $x \in \nu^{-1}(0,0,1)$

- maximum of f: local degree 1

- saddle : degree -1

– one maximum and k saddles in the preimage

- $\nu = 1-k$

- do horizontal embedding of Σ_k

- k minima

- $k+(k-1)$ sattles

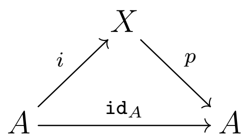
- $\deg(\nu) = k-(k+(k-1)) = 1-k$

3.8 (Deformation) retracts and quotients

A, X - topological spaces

- $i : A \to X$ - a map

Definition 3.54. *We say that A is a retract of X if there exists a map (retraction) $p : X \to A$ such that $p \circ i = \mathtt{id}_A$.*



Note: i is the inclusion of a subspace

- i is injective

- A has the induced topology

– U - open in A

– $p^{-1}(U)$ is open in X

– $U = i^{-1}(p^{-1}(U))$

Example: $i : * \to X$

- $*$ is retract of X, use $p : X \to *$

(H, ∂) - homology theory

$i : A \to X$

Corollary 3.55. *If A is a retract of X, then we have a decomposition*

$$H(X) \cong H(A) \oplus \textit{complement} .$$

Proof.

$p : X \to A$ such that $p \circ i = \mathtt{id}_A$

- $\pi := i_* p_* : H(X) \to H(X)$ is projection

– $\pi\pi = (i_* p_*)(i_* p_*) = i_*(p_* i_*)p_* = i_* p_* = \pi$

- $(1 - \pi)$ is auch projection: $(1 - \pi)(1 - \pi) = 1 - 2\pi + \pi^2 = 1 - 2\pi + \pi = 1 - \pi$

- $H(X) = \mathtt{im}(\pi) \oplus \mathtt{im}(1 - \pi)$

– $\pi(H(X)) = \mathtt{im}(i_*)$

— $\pi \circ i_* = i_* p_* i_* = i_*$ shows $\mathtt{im}(\pi) \subseteq \mathtt{im}(i_*)$

— $\mathtt{im}(\pi) = \mathtt{im}(i_* p_*) \subseteq \mathtt{im}(i_*)$

– $(1 - \pi(X)) = \mathtt{ker}(p_*)$

— $p_*(1 - \pi) = p_*(1 - i_* p_*) = p_* - p_* i_* p_* = p_* - p_* = 0$ shows $\mathtt{im}(1 - \pi) \subseteq \mathtt{ker}(p_*)$

— assume x in $\mathtt{ker}(p_*)$: $(1 - \pi)(x) = x - i_* p_*(x) = x$, hence $x \in \mathtt{im}(1 - \pi)$

- $i_* : H(A) \to \mathtt{im}(\pi)$ is isomorphism since i_* is injective.

□

note that complement depends on choice of p

consider retract

$i : A \to X$, $p : X \to A$, $i \circ p = \mathtt{id}_A$

- assume i is homotopy equivalence with inverse p

– $p \circ i = \mathtt{id}$

– $i \circ p \overset{H}{\sim} \mathtt{id}_X$

- homotopy $[0, 1] \times X \to X$ with $H(0, -) = \mathtt{id}$ and $H(1, -) = p \circ i$

consider quotient $\bar{i} : * \cong A/A \to X/A$, $\bar{p} : X/A \to A/A \cong *$

- have $\bar{p} \circ \bar{i} \cong \mathtt{id}_*$

Question:

- is $\bar{i} : A/A \to X/A$ still homotopy equivalence?

– in general H does not factorize over quotient

Definition 3.56. *H is called a strong (weak) deformation retraction if*

$$H \circ (\mathtt{id}_{[0,1]} \times i) = \mathtt{id}_{[0,1]\times A} \ , \quad (\ H([0,1] \times A) \subseteq A \ )$$

In this case we call A a strong (weak)deformation retract of X.

a weak deformation retraction induces a homotopy $\bar{H} : [0,1] \times X/A \to X/A$ from $\mathtt{id}_{X/A}$ to $\bar{i} \circ \bar{p}$.

Example:

$\{0\} \times D^n \cup [0,1] \times S^{n-1}$ in $[0,1] \times D^n$ is a strong deformation retract

- embed $[0,1] \times D^n$ into $\mathbb{R} \times \mathbb{R}^n$
- H moves along the rays from $(2,0)$
- picture

$i : A \to X$ inclusion of subspace

want to calculate homology of X/A

Proposition 3.57. *If A is a weak deformation retract of a neighbourhood in X, then we have a canonical isomorphism*

$$H(X/A, A/A) \cong H(X,A) \ .$$

preparation:

consider subspaces $A \subseteq B \subseteq X$

define sequence

$$H(B,A) \stackrel{\alpha}{\to} H(X,A) \stackrel{\beta}{\to} H(X,B) \stackrel{\delta}{\to} H(B,A)[-1] \tag{3.1}$$

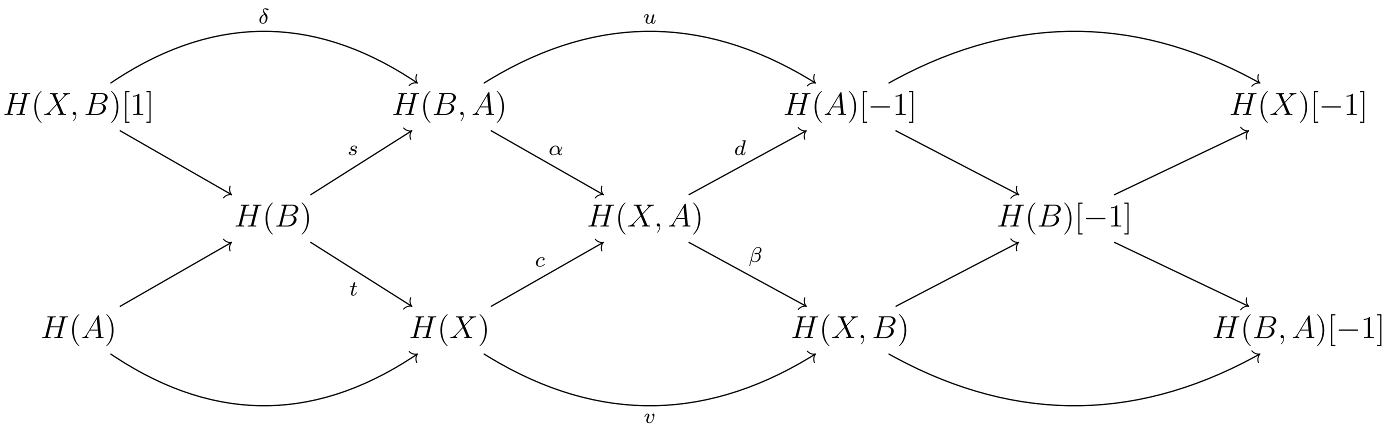
where

$$\delta : H(X,B) \stackrel{\partial_{(X,B)}}{\to} H(B)[-1] \to H(B,A)[-1]$$

- all other maps come from inclusions

Lemma 3.58. *The sequence* (3.1) *is exact.*

Proof. discuss with braid diagram



exactness at $H(X,A)$

$$\begin{array}{ccc} H(B,A) & \longrightarrow & H(B,B) \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ H(X,A) & \xrightarrow{\beta} & H(X,B) \end{array}$$

and $H(B,B) = 0$ shows $\beta\alpha = 0$

assume x in $H(X,A)$, $\beta(x) = 0$

- then exists y in $H(B,A)$ with $u(y) = d(x)$
- $d(x - \alpha(y)) = 0$ - find z in $H(X)$ with $c(z) = x - \alpha(y)$
- $v(z) = \beta(c(z)) = 0$
- find w in $H(B)$ with $t(w) = z$
- then $x = \alpha(s(w) + y)$

all other places similar (and easier)

□

Proof. (of Prop. 3.57) $A \to V \to X$

- A strong deformation retract of V
-

$$\begin{array}{ccccc} H(X,A) & \xrightarrow{\cong} & H(X,V) & \xleftarrow[exc]{\cong} & H(X\setminus A, V\setminus A) \\ \big\downarrow & & \big\downarrow & & \big\downarrow\cong \\ H(X/A,A/A) & \xrightarrow{\cong} & H(X/A,V/A) & \xleftarrow[exc]{\cong} & H(X/A\setminus A/A, V/A\setminus A/A)\end{array}$$

- right horizontal isomorphism: triple sequence and $H(V, A) = 0$ and $H(V/A, A/A) = 0$

– this uses assumption on weak deformation retract

- left horizontal isomorphism: excision (cut out A or A/A respectively)

– right vertical isomorphism: is induced by homeomorphism

□

application to reduced suspension:

$(X, *)$ - a pointed space

Definition 3.59. *$(X, *)$ is called well-pointed if $*$ is a strong deformation retract of a neighbourhood of $*$.*

Definition 3.60. *If $(X, *)$ is well-pointed, then $\tilde{H}(\tilde{\Sigma}X) \cong \tilde{H}(X)[-1]$.*

Proof. $(X, *)$ is well-pointed

- image of $[-1, 1] \times \{*\}$ in ΣX is strong deformation retract of neighbourhood

- $\tilde{H}(\tilde{\Sigma}X) \cong H(\tilde{\Sigma}X, *) \cong H(\Sigma X, [-1, 1] \times \{*\}) \cong H(\Sigma X, \{(0, *)\}) \cong \tilde{H}(\Sigma X) \cong \tilde{H}(X)[-1]$

□

3.9 CW-complexes

consider diagram $X : \mathbb{N} \to \mathbf{Top}$

- $X_0 \overset{\phi_0}{\to} X_1 \overset{\phi_1}{\to} X_2 \overset{\phi_2}{\to} \dots$

Definition 3.61. *We define telescope of X as coequalizer*

$$T(X) := \texttt{colim}\left(\bigsqcup_{n\in\mathbb{N}} X_n \underset{x_n\mapsto(n+1,\phi_n(x_n))}{\overset{x_n\mapsto(n+1,x_n)}{\rightrightarrows}} \bigsqcup_{n\in\mathbb{N}} [n, n+1] \times X_n \right)$$

$$T(X) := \bigsqcup_{n\in\mathbb{N}} [n, n+1] \times X_n / \sim \ , \quad (n+1, x_n) \sim (n+1, \phi_n(x_n))$$

picture

have natural map $c : T(X) \to X$, induced by $(u, x_n) \mapsto c_n(x_n)$

- here $c_n : X_n \to \mathtt{colim}_{\mathbb{N}} X$ is the canonical map

consider diagram $X : \mathbb{N} \to \mathbf{Top}$

(H, ∂) - homology theory

Lemma 3.62. *We have* $\mathtt{colim}_{\mathbb{N}} H(X) \cong H(T(X))$.

Proof.

decompose $T(X)$ into open subsets

$$A := ([0,1) \times X_0) \cup \bigsqcup_{n\in\mathbb{N}, n\geq 1} ((2n - 1/4, 2n) \times X_{2n-1} \cup [2n, 2n+1) \times X_{2n})$$

$$B := \bigsqcup_{n\in\mathbb{N}} ((2n+1-1/4, 2n+1) \times X_{2n} \cup [2n+1, 2n+2) \times X_{2n+1})$$

then

$$A \cap B = \bigsqcup_{n\in\mathbb{N}} (n+1-1/4, n+1) \times X_n$$

observe now

$$A \simeq \bigsqcup_{n\in\mathbb{N}} X_{2n}$$

$$B \simeq \bigsqcup_{n\in\mathbb{N}} X_{2n+1}$$

$$A \cap B \simeq \bigsqcup_{n\in\mathbb{N}} X_n$$

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H(A\cap B) & \longrightarrow & H(A)\oplus H(B) & \longrightarrow & H(T(X)) & \longrightarrow & \dots \\
 & & \downarrow\cong & & \downarrow\cong & & \| & & \\
\dots & \longrightarrow & \bigoplus_{n\in\mathbb{N}} H(X_n) & \xrightarrow{\alpha} & \bigoplus_{n\in\mathbb{N}} H(X_n) & \longrightarrow & H(T(X)) & \longrightarrow & \dots \\
 & & {\scriptstyle x_n\mapsto(-1)^n x_n}\downarrow\cong & & \downarrow\cong & & \| & & \\
\dots & \longrightarrow & \bigoplus_{n\in\mathbb{N}} H(X_n) & \xrightarrow{\beta} & \bigoplus_{n\in\mathbb{N}} H(X_n) & \longrightarrow & H(T(X)) & \longrightarrow & \dots
\end{array}$$

- structure map
- $\alpha(x_n) = (-1)^n x_n - (-1)^n \phi_n(x_n)$
- $\beta(x_n) = x_n - \phi_n(x_n)$

claim β is injective:

- $\beta(\sum_n x_n) = 0$ implies
- $x_0 = 0$
- $x_{n+1} = \phi_n(x_n)$ for all $n \geq 0$

– conclude $x_n = 0$ for all n in $\mathbb{N}$

- conclude: $H(T(X)) \cong \operatorname{coker}(\beta)$

- $\operatorname{coker}(\beta) \cong \mathtt{colim}_{n\in\mathbb{N}} H(X_n)$

– Exercise

□

(X, A) - pair of topological spaces

Definition 3.63. *A relative CW-complex structure on (X, A) is an increasing filtration*

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X$$

von X by subspaces such that:

1. $X \cong \mathtt{colim}_{\mathbb{N}} X_n$
2. *For every n in $\mathbb{N}$ there exists a set I_n and a push-out diagram*

$$\begin{array}{ccc}
\bigsqcup_{I_n} S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\bigsqcup_{I_n} D^n & \longrightarrow & X_n
\end{array} .$$

(here we set $S^{-1} := \emptyset$))

Definition 3.64. *A CW-complex is a space X together with a relative CW-complex structure on $(X, \emptyset)$.*

Remark:

$X_n \setminus X_{n-1} \cong \coprod_{e \in I_n} (D^n \setminus S^{n-1})$

- hence I_n is set of connected components of $X_n \setminus X_{n-1}$
- it is determined by the CVW-structure
- the components are called the open cells
- write $\sqcup_{e \in I_n} \chi_e : \bigsqcup_{e \in I_n} S^{n-1} \to X_{n-1}$

– χ_e is called the attaching map of the cell e

- write $\sqcup_{e \in I_n} \tilde{\chi}_e : \bigsqcup_{e \in I_n} D^n \to X_n$

– $\tilde{\chi}_e$ is called the characteristic map of the cell e

– have map of pairs $\tilde{\chi}_e : (D^n, S^{n-1}) \to (X_n, X_{n-1})$

Beispiele: $\mathbb{R}^n$, S^n, $\mathbb{CP}^n$, $\mathbb{RP}^n$

(X, A) - a pair of spaces

Lemma 3.65. *If (X, A) is a relative CW-complex, then the subspace $(\{0\} \times X) \cup ([0,1] \times A)$ is a deformation retract of $[0,1] \times X$.*

Proof.

- $(\{0\} \times D^n) \cup ([0,1] \times S^{n-1})$ in $[0,1] \times D^n$ is a strong deformation retract
- apply strong deformation retraction to the n-discs attached to X_{n-1}
- get strong deformation retraction $H_n : [0,1] \times X_n \to X_n$ of $[0,1] \times X_n$ to $(\{0\} \times X_n) \cup ([0,1] \times X_{n-1})$
- concatenate strong deformation retractions for $n = 0, 1, \ldots$ such that H_n uses time interval $[1 - 1/2^n, 1 - 1/2^{n+1}]$
- for point in X_n retraction is constant for times $\geq 1 - 1/2^{n+1}$ (here strong is important)
- infinite concatenation is continuous

□

(X, A) - a relative CW-complex

- get diagram $A = X_{-1} \to X_0 \to X_1 \to X_2 \to \dots$

- $T(X)$ - telescope of this diagram embedded in $[-1, \infty) \times X$

Lemma 3.66. *$c : (T(X), A) \to (X, A)$ is a homotopy equivalence.*

Proof.

- $X_n \to X$ - inclusions of closed subspace

- define map $i : T(X) \to [-1, \infty) \times X$ by $(t, x_n) \mapsto (t, x_n)$

- consider factorization of canonical map $c : T(X) \xrightarrow{i} [0, \infty) \times X \xrightarrow{\mathtt{pr}_X} X$

– $\mathtt{pr}_X$ is homotopy equivalence

– it suffices to show that i is homotopy equivalence

- will actually show that $T(X)$ is strong deformation retract of $[-1, \infty) \times X$

– $Y_i := T(X) \cup ([i, \infty) \times X)$

– claim: Y_i strongly deformation retracts on Y_{i+1}

– (X, X_i) is a CW-pair

– $[i, i + 1] \times X$ strongly deformation retracts into $(\{i + 1\} \times X) \cup ([i, i + 1] \times X_i)$

– $T(X) \cap ([i, i + 1] \times X) = ([i, i + 1] \times X_i) \cup (\{i + 1\} \times X_{i+1})$

– extend this strong deformation retraction by identity on $(T(X) \cap ([-1, i] \times X)) \cup ([i + 1, \infty) \times X)$

- concatenate all these deformation retractions so that on Y_i time interval $[1-2^{-i}, 1-2^{-i-1}]$ is used

– for (t, x) in $[i, \infty) \times X_i$ deformation retraction is constant for $t > 2^{-i-1}$ (use strongness)

- infinite concatenation is continuous

□

Corollary 3.67. *If (X, A) is a relative CW-complex, then $H(X, A) \cong \mathtt{colim}_{n\in\mathbb{N}} H(X_n, A)$.*

Proof.

- use $H(T(X), A) \cong H(X, A)$ by Lemma 3.66

- use $H(T(X), A) \cong \mathtt{colim}_{\mathbb{N}} H(X_n, A)$ by Lemma 3.62

□

$(X_n, *_i)_{n\in I}$ - family of well-pointed spaces

- form

$$\bigvee_{i\in\mathbb{N}} X_i := \bigsqcup_{i\in I} X_i/\sim$$

where $\sim$ identifies all base points to one point $*$

Lemma 3.68. $H(\bigvee_{i\in I} X_i, *) \cong \bigoplus_{i\in I} H(X_i, *_i)$.

Proof.

use well-pointedness: find U_i - neighbourhood of $*_i$ deformation retracting on $*_i$

- $U := \bigcup_{i\in I} U_i/\sim$ open neighbourhood of $*$ deformation retracting on $*$

$$\begin{aligned} H(\bigvee_{i\in I} X_i, *) &\overset{htpy}{\cong} H(\bigvee_{i\in I} X_i, U) \\ &\overset{exc}{\cong} H(\bigvee_{i\in I} X_i \setminus \{*\}, U\setminus\{*\}) \\ &\overset{homeo}{\cong} H(\bigsqcup_{i\in I} X_i\setminus\{*_i\}, U_i\setminus\{*_i\}) \\ &\overset{add}{\cong} \bigoplus_{i\in I} H(X_i\setminus\{*_i\}, U_i\setminus\{*_i\}) \\ &\overset{exc}{\cong} \bigoplus_{i\in I} H(X_i, U_i) \\ &\overset{htpy}{\cong} \bigoplus_{i\in I} H(X_i, *_i) \end{aligned}$$

□

standing assumptions:

- assume $H(*) \cong \mathbb{Z}[0]$
- for every pair (X, A) we have $H_k(X, A) \cong 0$ for $k < 0$

(X, A) - relative CW-complex

- I_n - set of n-cells

set $C_n(X, A) := H_n(X_n, X_{n-1})$

- for e in I_n have map $i_e : \mathbb{Z} \cong H_n(D^n, S^{n-1}) \overset{\tilde{\chi}_{e,*}}{\to} H_n(X_n, X_{n-1}) = C_n(X, A)$

Lemma 3.69. *The collection of these maps induce an isomorphism* $\bigoplus_{e\in I_n} \mathbb{Z} \cong C_n(X, A)$.

Proof.

- construct open neighbourhood $\tilde{X}_{n-1}$ of X_{n-1} in X_n such that $X_{n-1} \to \tilde{X}_{n-1}$ is strong deformation retract

– define $\tilde{X}_{n-1}$ by push-out

$$\begin{array}{ccc} \bigsqcup_{I_n} S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \bigsqcup_{I_n} (D^n \setminus 1/2D^n) & \longrightarrow & \tilde{X}_{n-1} \end{array} .$$

– inclusion $S^{n-1} \to (D^n \setminus 1/2D^n)$ is strong deformation retract

– can glue: $X_{n-1} \to \tilde{X}_{n-1}$ strong deformation retract

– $H(X_n, X_{n-1}) \cong H(X_n/X_{n-1}, *)$

– observe $X_n/X_{n-1} \cong \bigvee_{I_n} S^n$

– $H_n(X_n, X_{n-1}) \cong H_n(\bigvee_{I_n} S^n, *) \cong \bigoplus_{I_n} \mathbb{Z}$

□

define $\partial : C_n(X, A) \to C_{n-1}(X, A)$ by

$$C_n(X, A) = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}, A) \to H_{n-1}(X_{n-1}, X_{n-2})$$

Lemma 3.70. *The composition $C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A)$ vanishes.*

Proof. the composition expands as

$$C_{n+1}(X, A) \xrightarrow{\partial} H_n(X_n, A) \to H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}, A) \to C_{n-1}(X, A)$$

- middle composition vanishes by exactness of sequence for triple (X_n, X_{n-1}, A) □

Definition 3.71. *$(C(X, A), \partial)$ is called the cellular chain complex of (X, A).*

(X, Y), (X', A') - relative CW-complexes

$f : (X, A) \to (X', A')$ - a map of pairs

Definition 3.72. *f is called cellular, if $f(X_n) \subseteq X'_n$ for all n in $\mathbb{N}$.*

Lemma 3.73. *A cellular map induces a chain map $f_* : C(X, A) \to C(X', A')$.*

Proof.

f induces map of pairs $(X_n, X_{n-1}) \to (X'_n, X'_{n-1})$ for all n in $\mathbb{N}$

- hence map $f_* : C_n(X, A) \to C_n(X', A')$

f_* is chain map by commutativity of

$$\begin{array}{ccccc} H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}, A) & \longrightarrow & H_{n-1}(X_{n-1}, X_{n-2}) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_n(X'_n, X'_{n-1}) & \xrightarrow{\partial'} & H_{n-1}(X'_{n-1}, A') & \longrightarrow & H_{n-1}(X'_{n-1}, X'_{n-2}) \end{array}$$

□

Lemma 3.74. *We have $H(X, A) \cong H(C(X, A), \partial)$.*

Proof.

consider map $H_k(X_n, A) \to H_k(X_{n+1}, A)$

do induction by n for fixed k and then by k

intermediate claims $Claim(k, n)$

Claim(k,n) $H_k(X_n, A) = 0$ for $n < k$

Claim(k,k) $H_k(X_k, A) \cong \mathtt{ker}(\partial : C_k(X, A) \to C_{k-1}(X, A))$

Claim(k,k+1) $H_k(X_{k+1}, A) \cong \frac{\mathtt{ker}(\partial : C_k(X,A) \to C_{k-1}(X,A))}{\mathtt{im}(\partial : C_{k+1}(X,A) \to C_k(X,A))}$

Claim(k,n) $H_k(X_{n-1}, A) \xrightarrow{\cong} H_k(X_n, A)$ for $n > k + 1$

assertions clear for $k = -1$ and all n

fix k and assume that $Claim(k - 1, n)$ has been shown for all n

start now induction by n with $n = -1$

- then claim $Claim(k, n)$ is true

as long as $k > n$ show that $Claim(k, n - 1)$ implies $Claim(k, n)$

- long exact sequence for (X_n, X_{n-1}, A)

- $H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}, A) \to H_k(X_n, A) \to H_k(X_n, X_{n-1})$

– both outer terms zero

– $0 \overset{Claim(k,n-1)}{=} H_k(X_{n-1}, A) \cong H_k(X_n, A)$ shows Claim (k, n)

– use long exact sequence for (X_k, X_{k-1}, A) - upper line exact

– use $Claim(k-1, k-1)$ for injectivity as indicated

-

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_k(X_k, A) & \xrightarrow{!} & H_k(X_k, X_{k-1}) & \longrightarrow & H_{k-1}(X_{k-1}, A) \\ & & & & & \searrow^{\partial} & \big\downarrow \\ & & & & & & H_{k-1}(X_{k-1}, X_{k-2}) \end{array}$$

is exact

- get injectivity of marked arrow, hence $Claim(k, k)$

– use long exact sequence for (X_{k+1}, X_k, A) - upper line exact

– use $Claim(k, k)$ for injectivity

-

$$\begin{array}{ccccccc} H_{k+1}(X_{k+1}, X_k) & \longrightarrow & H_k(X_k, A) & \longrightarrow & H_k(X_{k+1}, A) & \longrightarrow & 0 \\ & \searrow^{\partial} & \big\downarrow & & & & \\ & & H_k(X_k, X_{k-1}) & & & & \end{array}$$

is exact

get $Claim(k, k+1)$

- use long exact sequence for (X_n, X_{n-1}, A)

- $H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}, A) \to H_k(X_n, A) \to H_k(X_n, X_{n-1})$

- outer terms zero if $n > k+1$

- $H_k(X_{n-1}, A) \overset{\cong}{\to} H_k(X_n, A)$ for all $n > k+1$

- hence $Claim(k, n)$ fo $n > k+1$

this finishes induction in n for fixed k

- now increase k

finally use Corollary 3.67

$$H_k(X, A) \cong \operatorname*{\texttt{colim}}_{n \in \mathbb{N}} H_k(X_n, A)$$

□

calculation of $\partial : C_n(X,A) \to C_{n-1}(X,A)$

choose attaching maps χ_e for e in I_e and χ_f for f in I_{n-1}

- e in I_n

– $i_e : \mathbb{Z} \to C_n(X,A)$

- $i_e(1) =: [e]$ basis element in $H_n(X_n, X_{n-1}) = C_n(X,A)$

- f in I_{n-1}

- need geometric picture of projection $p_f : C_{n-1}(X,A) \to \mathbb{Z}$ to summand $\mathbb{Z}[f]$

- introduce notation $X_{n-1}^{-f} := X_{n-2} \cup \bigcup_{f' \in I_{n-1} \setminus \{f'\}} \tilde{\chi}_f(D^n)$

- $(D^{n-1}, S^{n-2}) \stackrel{\tilde{\chi}_{f''}}{\to} (X_{n-1}, X_{n-2}) \to (X_{n-1}, X_{n-1}^{-f}) \stackrel{\tilde{\chi}_f}{\leftarrow} (D^{n-1}, S^{n-1})$

- $n = 1$

– $* \stackrel{\tilde{\chi}_{f''}}{\to} X_0 \to X_0/X_0^{-f} \cong *_A \sqcup *_f$ with $*_f := \tilde{\chi}_f(*)$

- $n \geq 2$

- $S^{n-1} \cong D^{n-1}/S^{n-2} \stackrel{\tilde{\chi}_{f''}}{\to} X_{n-1}/X_{n-2} \to X_{n-1}/X_{n-1}^{-f} \stackrel{\tilde{\chi}_f}{\cong} D^{n-1}/S^{n-2} \cong S^{n-1}$
is identity for $f'' = f$ and constant else

- $n = 1$

$p_f : C_0(X,A) = H_0(X_0, A) \to H_0(X_0/X_0^{-f}, *_A) \cong H_0(*_f) \cong \mathbb{Z}$

- $n \geq 2$

$p_f : C_{n-1}(X,A) = H_{n-1}(X_{n-1}, X_{n-2}) \cong H_{n-1}(X_{n-1}/X_{n-2}, *) \to H_{n-1}(X_{n-1}/X_{n-1}^{-f}, *) \cong H_{n-1}(S^{n-1})$

this map is projection onto desired summand

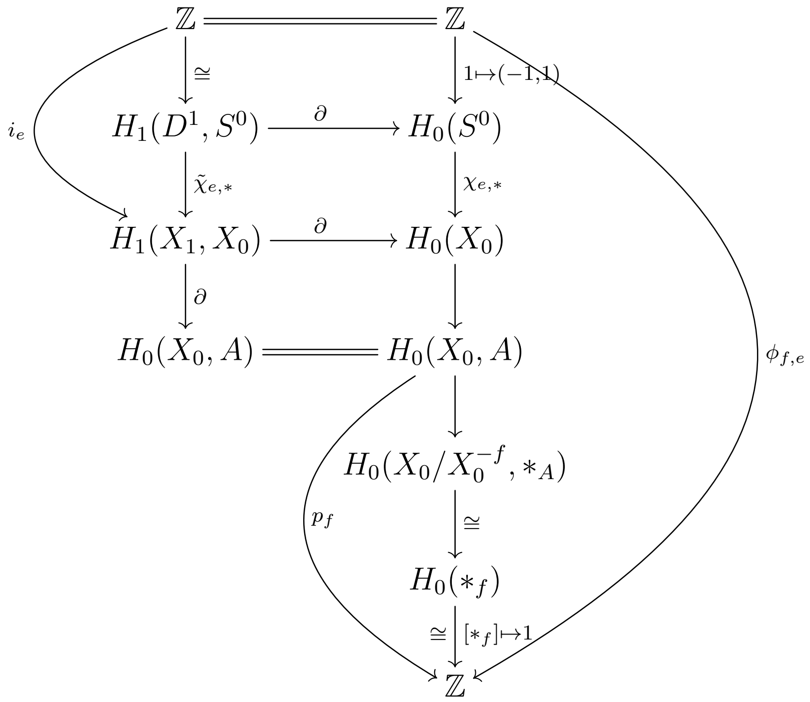
define $\phi_{f,e}$ by

$$\begin{array}{ccc} C_n(X,A) & \xrightarrow{\partial} & C_{n-1}(X,A) \\ {\scriptstyle i_e}\uparrow & & \downarrow{\scriptstyle p_f} \\ \mathbb{Z} & \xrightarrow{\phi_{f,e}} & \mathbb{Z} \end{array}$$

must calculate $\phi_{f,e}$

$n = 1$

$\psi_{f,e} : S^0 \overset{\chi_e}{\to} X_0 \to X_0/X_0^{-f} \cong *_A \sqcup *_f$

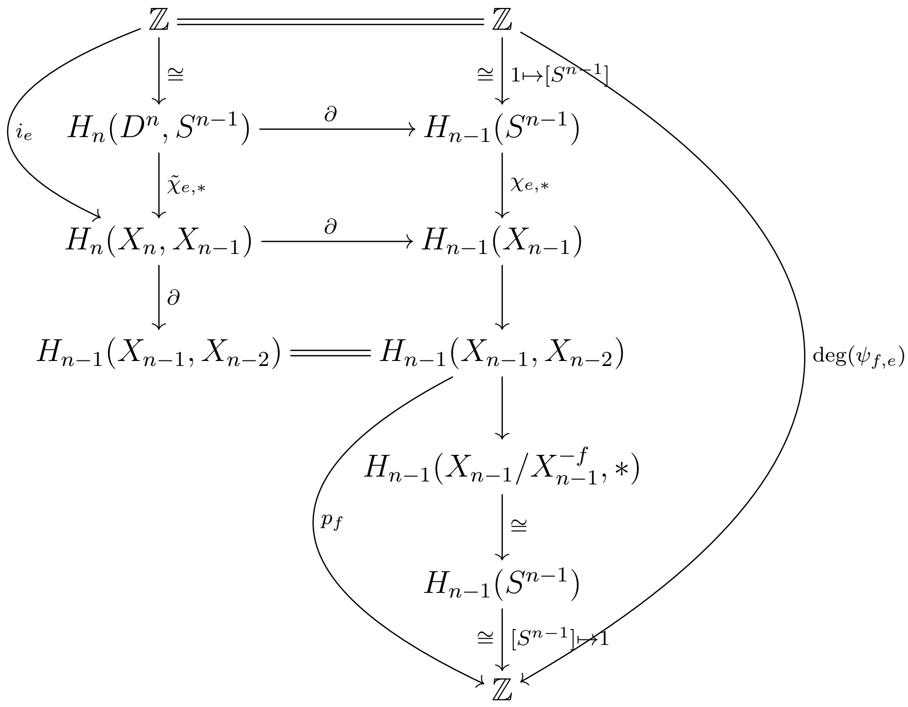


conclude:

$$\phi_{f,e} = \begin{cases} -1 & \chi_e(-1) = f \& \chi_e(1) \neq f \\ 1 & \chi_e(1) = f \& \chi_e(-1) \neq f \\ 0 & else \end{cases}$$

$n \geq 2$

$$\psi_{f,e} : S^{n-1} \overset{\chi_e}{\to} X_{n-1} \to X_{n-1}/X_{n-1}^{-f} \overset{\tilde{\chi}_f}{\cong} S^{n-1}$$



$$\phi_{f,e} = \deg(\psi_{f,e})$$

3.10 Calculations

X - a CW-complex

$\pi_0(X)$ is set of path components

have canonical map $I_0 \to \pi_0(X)$, $e \mapsto [\chi_e(*)]$

- extends to $\tilde{h} : C_0(X) \to \mathbb{Z}[\pi_0(X)]$

Lemma 3.75. *The map $\tilde{h}$ factorizes over an isomorphism $h : H_0(X) \xrightarrow{\cong} \mathbb{Z}[\pi_0(X)]$*

Proof.

canonical map $I_0 \to \pi_0(X)$ is surjective (exercise)

- induced map $C_0(X) \to \mathbb{Z}[\pi_0(X)]$ is surjective

– formula: $\sum_{e \in I_0} m_e[e] \mapsto \sum_{c \in \pi_0(X)} (\sum_{e \in I_0, e \in c} m_e) c$

show that factorizes over $H_0(X)$

- consider $\sum_{f \in I_1} m_f f \in C_1(X)$

- $\delta(f) = [\chi_f(1)] - [\chi_f(-1)]$ in $C_0(X)$

- $\delta(\sum_{f \in I_1} m_f f) = \sum_{f \in I_1} m_f([\chi_f(1)] - [\chi_f(-1)]) = \sum_{e \in I_0} \left(\sum_{f \in I_1, \chi_f(1)=e} m_f - \sum_{f \in I_1, \chi_f(-1)=e} m_f \right)$

- consider path component c

- $\sum_{e \in I_0, e \in c} \left(\sum_{f \in I_1, \chi_f(1)=e} m_f - \sum_{f \in I_1, \chi_f(-1)=e} m_f \right) = \left(\sum_{f \in I_1, \chi_f(1) \in c} m_f - \sum_{f \in I_1, \chi_f(-1) \in c} m_f \right) = 0$

since for all f in I_1: $\chi_f(-1) \in c$ iff $\chi_f(1) \in c$

get surjection

$H_0(X) \to \mathbb{Z}[\pi_0(X)]$

now observe: $\pi_0(X_1) \to \pi_0(X)$ is a bijection

for every c in π_0 fix base point e_c in I_0 such that $\chi_{e_c}(*) \in c$

- for every e in I_0 with $\chi_e(*) \in c$ choose path from $\chi_e(e_c)$ to $\chi_e(*)$ as concatenation of images of 1-cells $f_n, f_{n-1}, \ldots, f_1$

- set $x_e := \sum_{i=1}^n [f_i]$ in $C_1(X)$

- then $\delta(x_e) = [e] - [e_c]$

$x = \sum_{e \in I_0} m_e[e]$

- $x \mapsto 0$

- this means for all c in $\pi_0(X)$

- $\sum_{e \in I_0, e \in c} m_e = 0$

- set $y := \sum_{e \in I_0} m_e x_e$

- $\delta(y) = \sum_{e \in I_0} m_e \delta(x_e) = \sum_{e \in I_0} m_e[e] - \sum_{c \in \pi_0(X)} \left(\sum_{e \in I_0, e \in c} m_e \right) [e_c] = x$

- hence $[x] = 0$ in $H_0(X)$

$\square$

$\mathbb{CP}^n$

- $|I_{2n}| = 1$

- $|I_{2n+1}| = 0$

- $C(\mathbb{CP}^n)$:

$$\mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \dots \mathbb{Z}$$

- ends in degree $2n$

$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k \text{ even and } 0 \leq k \leq 2n \\ 0 & else \end{cases}$

generator $[\mathbb{CP}^n] \in H_{2n}(\mathbb{CP}^n)$ is fundamental class

- x in interior of $2n$-cell

$\mathbb{CP}^n_{2n-1} \cong \mathbb{CP}^{n-1}$ - $2n-1$-skeleton

- $H_{2n}(\mathbb{CP}^n) \cong H_{2n}(\mathbb{CP}^n, \mathbb{CP}^n_{2n-1}) \to H_{2n}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \{x\}) \cong H_{2n}(D^{2n}, D^{2n} \setminus \{x\}) \cong H_{2n}(D^{2n}, S^{2n-1})$

via characteristic map, homotopy invariance and excision

- sends generator to generator

$\mathbb{CP}^\infty := \mathtt{colim}_n \, \mathbb{CP}^n$

$H_k(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & else \end{cases}$

$\mathbb{RP}^n$

$|I_k| = 1$ for $k = 0, 1, \dots, n$

$\mathbb{Z} \overset{d_1}{\leftarrow} \mathbb{Z} \overset{d_2}{\leftarrow} \mathbb{Z} \overset{d_3}{\leftarrow} \dots \mathbb{Z} \overset{d_n}{\leftarrow} \mathbb{Z}$

clear: $d_1 = 0$

$d_k = \deg(S^k \to \mathbb{RP}^k \to \mathbb{RP}^k/\mathbb{RP}^{k-1} \cong S^k)$

exercise: $d_k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$\mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{2}{\leftarrow} \mathbb{Z} \overset{0}{\leftarrow} \dots \mathbb{Z} \overset{d_n}{\leftarrow} \mathbb{Z}$ ends in degree n

n - even

$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \text{ even or } k > n \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \end{cases}$

n - odd

$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \text{ even or } k > n \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd, } k = 1, 3, \dots, n-2 \end{cases}$

if n is odd: generator $[\mathbb{RP}^n]$ in $H_n(\mathbb{RP}^n)$ is fundamental class

$\mathbb{RP}^\infty := \mathtt{colim}_n \mathbb{RP}^n$

$$H_k(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \end{cases}$$

Moore spaces

- fix k in $\mathbb{Z}$
- consider map $f : S^n \to S^n$ of degree k

– to be specific: use standard map $u \mapsto u^k$ for S^1 and its $n-1$-fold suspension for f

let $n \geq 1$, $k \neq 0$

Definition 3.76. *If $k \neq 0$, then we define the Moore space $M^n(k)$ as the push-out*

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ \downarrow & & \downarrow \\ D^{n+1} & \longrightarrow & M^n(k) \end{array} .$$

We set $M^n(0) := S^n$.

assume $k \neq 0$:

$M^n(k)$ is a $n+1$-dimensional CW-complex

- $* \subseteq S^n \subseteq M^n(k)$
- chain complex $C(M^n(k))$
- $\mathbb{Z} \xleftarrow{0} \cdots \leftarrow \mathbb{Z} \xleftarrow{f_*} \mathbb{Z}$
- last $\mathbb{Z}$ in degree $n+1$

$$H_\ell(M^n(k)) \cong \begin{cases} \mathbb{Z} & \ell = 0 \\ \mathbb{Z}/k\mathbb{Z} & \ell = n \\ 0 & else \end{cases}$$

Corollary 3.77. *Let $(I_n)_{n\geq 1}$ be a family of sets and for every n in $\mathbb{N}$ $(k_{n,i})_{i\in I_n}$ be a family of integers. Then there exists a pointed space X such that*

$$H_n(X, *) \cong \bigoplus_{i\in I_n} \mathbb{Z}/k_{n,i}\mathbb{Z}$$

for all n in $\mathbb{N}$ with $n \geq 1$.

Proof. Take $X := \bigvee_{n\in\mathbb{N}} \bigvee_{i\in I_n} M^n(k_{n,i})$. ☐

Question: let Y be a space

- assume $H_\ell(Y) \cong \bigoplus_{i\in I_n} \mathbb{Z}/k_{n,i}\mathbb{Z}$
- is Y homotopy equivalent to the corresponding wedge of Moore spaces?

(X, A) - relative CW-complex

Definition 3.78.

1. *(X, A) is locally finite if I_n is finite for every $n \in \mathbb{N}$.*
2. *The dimension of (X, A) is defined by* $\dim(X, A) := \max\{n \in \mathbb{N} \mid I_n \neq \emptyset\}$.
3. *(X, A) is finite if it has finitely many cells.*

note: locally finite and finite-dimensional is equivalent to finite

- $H(*) \cong \mathbb{Z}[0]$

Corollary 3.79.

1. *If X is locally finite, then $H_\ell(X, A)$ is finitely generated for every ℓ in $\mathbb{Z}$.*
2. *We have $H_\ell(X, A) \cong 0$ for $\ell \geq \dim(X, A) + 1$.*

Proof.

1.

$H_\ell(X, A)$ is a subquotient of a finitely generated abelian group $C_\ell(X, A)$ and hence finitely generated

2.

$C_\ell(X, A) = 0$ for $\ell \geq \dim(X, A) + 1$

☐

A - abelian group

Definition 3.80. *Then rank of A is defined by*

$$\mathtt{rk}(A) := \sup\{n \in \mathbb{N} \mid \textit{there exists injective homomorphism } \mathbb{Z}^n \to A\} .$$

- $\mathtt{rk}(A) = \dim_{\mathbb{Q}} A \otimes \mathbb{Q}$

- if A is finitely generated, then $A \cong \mathtt{Tor}(A) \oplus \mathbb{Z}^{\mathtt{rk}(A)}$

- if A is torsion, then $\mathtt{rk}(A) = 0$

(X, A) - pair of spaces

- $H(*) \cong \mathbb{Z}[0]$

Definition 3.81. *The number* $b_\ell(X, A) := \mathtt{rk} H_\ell(X, A)$ *is called the* ℓ*'th Betti number of* (X, A).

- if (X, A) is locally finite CW-complex , then $b_\ell(X, A) < \infty$

(X, A) - pair of spaces

Definition 3.82. *Assume:*

1. *For every* ℓ *in* $\mathbb{N}$ *we have* $b_\ell(X, A) < \infty$.
2. *We have* $b_\ell(X, A) = 0$ *for* $\ell >> 0$.

Then the Euler characteristic of (X, A) *is defined by*

$$\chi(X, A) := \sum_{\ell \in \mathbb{Z}} (-1)^\ell b_\ell(X, A) \ .$$

- if (X, A) is finite relative CW-complex, then $\chi(X, A)$ is defined

(C, ∂) - finite chain complex of finite-dimensional $\mathbb{Q}$-vector spaces

Lemma 3.83. *We have the equlity*

$$\sum_{\ell \in \mathbb{Z}} (-1)^\ell \dim_{\mathbb{Q}}(H_\ell(C, \partial)) = \sum_{\ell \in \mathbb{Z}} \dim_{\mathbb{Q}}(C_\ell) \ .$$

Proof.

set

- $Z_\ell := \mathtt{ker}(\partial : C_\ell \to C_{\ell-1})$

- $B_\ell := \mathtt{im}(\partial : C_{\ell+1} \to C_\ell$

- note: $H_\ell := H_\ell(C, \partial) \cong Z_\ell / B_\ell$

- $C_\ell / Z_\ell \overset{\partial, \cong}{\to} B_{\ell+1}$

for vector spaces: subspaces have a complement have isomorphisms

$Z_\ell \cong B_\ell \oplus H_\ell$

$C_\ell \cong B_\ell \oplus H_\ell \oplus B_{\ell+1}$

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}}(-1)^\ell \dim_{\mathbb{Q}}(C_\ell) &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell(\dim_{\mathbb{Q}}(B_\ell)+\dim_{\mathbb{Q}}(H_\ell)+\dim_{\mathbb{Q}}(B_{\ell+1}))\\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \dim_{\mathbb{Q}}(H_\ell)\end{aligned}$$

□

Lemma 3.84. *If (X,A) is a finite relative CW-complex, then*

$$\chi(X,A)=\sum_{\ell\in\mathbb{N}}(-1)^\ell|I_\ell| \ .$$

Proof.

$\mathbb{Q}$ is flat over $\mathbb{Z}$

- for any chain complex C over $\mathbb{Z}$: $H(C)\otimes\mathbb{Q}\cong H(C\otimes\mathbb{Q})$

$$\begin{aligned}b_\ell(X,A) &= \dim_{\mathbb{Q}} H_\ell(X,A)\otimes\mathbb{Q}\\ &= \dim_{\mathbb{Q}} H_\ell(C(X,A))\otimes\mathbb{Q}\\ &= \dim_{\mathbb{Q}} H_\ell(C(X,A)\otimes\mathbb{Q})\\ &= \dim_{\mathbb{Q}} H_\ell(C(X,A;\mathbb{Q}))\end{aligned}$$

where $C(X,A;\mathbb{Q}) := C(X,A)\otimes\mathbb{Q}$

$$\begin{aligned}\chi(X,A) &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell b_\ell(X,A)\\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \dim_{\mathbb{Q}} H_\ell(C(X,A;\mathbb{Q}))\\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \dim_{\mathbb{Q}} C_\ell(X,A;\mathbb{Q})\\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell|I_\ell|\end{aligned}$$

□

Examples:

$\chi(S^{2n}) = 2$

$\chi(S^{2n+1}) = 0$

$\chi(\Sigma_k) = 2 - 2k$

$\chi(\mathbb{CP}^n) = n + 1$

$\chi(\mathbb{RP}^{2n}) \cong 1$

$\chi(\mathbb{RP}^{2n+1}) \cong 0$

$\chi(M^n(k)) = 1$ for $k \neq 0$

3.11 Applications to sections of tangent bundle

M in $\mathbb{R}^{n+1}$ immersed oriented n-submanifold

- $[M]$ - fundamental class, homological orientation

N in $\Gamma(M, \mathcal{N})$ - unit normal vector field, outward-pointing

$N : M \to S^n$ - Gauß map

Lemma 3.85. *Assume:*

1. *n is even.*
2. $\deg(N) \neq 0$

Then TM does not admit a nowhere vanishing section.

Proof.

- assume there is $X \in \Gamma(M, TM)$ - unit vector field

- get homotopy of maps
- $H : [0,1] \times M \to S^n$

– $H_t(x) := \cos(\pi t)N(x) + \sin(\pi t)X(x)$

— $H_0 = N$ (Gauß map)

— $H_1 = -N = a \circ N$ ($a : S^n \to S^n$ - antipodal map)

– $\deg(a) = -1$

– $\deg(N) = \deg(a \circ N) = -\deg(N)$

- hence $0 = \deg(N)$

□

Corollary 3.86. *Assume that $n \geq 1$. Then TS^n admits a nowhere vanishing section iff n is even.*

Proof.

1.

use standard embedding $S^n \to \mathbb{R}^{n+1}$

degree of Gauss map is 1

- no nowhere vanishing section for even n

2.

- $n = 2m - 1$

consider $U_t := \mathtt{diag}(e^{it}, \dots e^{it})$ in $U(m)$

- acts on S^n (as submanifold of $\mathbb{C}^m \cong \mathbb{R}^n$)
- define X in $\Gamma(S^n, TS^n)$ by

– $X(x) := (\partial_t)_{|t=0} U_t(x) := (ix, \dots, ix)$ for x in S^n

- this is nowhere vanishing

□

closed oriented surface Σ_k

Lemma 3.87.

1. *If $k \neq 1$, then $T\Sigma_k$ does not admits a nowhere vanishing section.*
2. *$T\Sigma_1$.*

Proof.

- can choose embedding $\Sigma_k \to \mathbb{R}^3$ such that degree of Gauß map is $1 - k$
- hence for $k \neq 1$ we know that $T\Sigma_k$ does not admits a nowhere vanishing section

$T\Sigma_1 = TT^2$ is trivial

□

4 Construction of homology theories

4.1 Simplicial objects

$[n] = \{0 < 1 < \cdots < n\}$ - a poset

Δ - category

- objects: $[n]$, $n = 0, 1, 2 \ldots$

- morphisms: order-preserving maps

Example:

- $d_i : [n] \to [n+1]$, $(0, \ldots, n) \mapsto (0, \ldots, i-1, i+1, \ldots, n+1)$ (table of values)

– i'th boundary

- $s_i : [n] \to [n-1]$, $(0, \ldots, n) \mapsto (0, \ldots, i, i, \ldots, n-1)$

– i'th degeneration

C - category

$c\mathbf{C} := \mathbf{Fun}(\Delta, \mathbf{C})$ - category of cosimplicial objects in **C**

Example of a cosimplicial space :

functor: $|-| : \Delta \to \mathbf{Top}$ in $c\mathbf{Top}$

- on objects: $|[n]| :=$ space of probability measures on $[n]$

- on morphisms: $f : [n] \to [m]$ induces $f_* : |[n]| \to |[m]|$ - push-forward of measures

- $|[n]| \cong \{(t_0, \ldots, t_n) \in [0,1]^{n+1} \mid \sum_{i=0}^n t_i = 1\} \subseteq [0,1]^{n+1}$

– $(t_0, \ldots, t_n)$ corresponds to measure $\sum_{i \in [n]} t_i \delta_i$

– $d_{i,*}(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$

– $s_{i,*}(t_0, \ldots, t_n) = (t_0, \ldots, t_i + t_{i+1}, t_{i+2}, \ldots, t_n)$

make pictures in dimensions ≤ 2

write also $\Delta^n := |[n]|$

here is a useful parametrization of the n-simplex by n-tuples of numbers

$$(0 \leq \phi_1 \leq \cdots \leq \phi_n \leq 1)$$

set

$\phi_1 := t_0$

$\phi_2 := t_0 + t_1$

$\vdots$

$\phi_i := t_0 + \cdots + t_{i-1}$

- set $\phi_0 := 0$ and $\phi_{n+1} := 1$
- get $t_i := \phi_{i+1} - \phi_i$
- $d_{i,*}(\phi_1, \ldots, \phi_n) = \begin{cases} (\phi_1, \ldots, \phi_i, \phi_i, \phi_{i+1}, \ldots, \phi_n) & 1 \le i \le n \\ (0, \phi_1, \ldots, \phi_n) & i = 0 \\ (\phi_1, \ldots, \phi_n, 1) & i = n+1 \end{cases}$
- $s_{i,*}(\phi_1, \ldots, \phi_n) = (\phi_1, \ldots, \hat{\phi}_{i-1}, \ldots \phi_n)$

$s\mathbf{C} := \mathbf{Fun}(\Delta^{op}, \mathbf{C})$ - category of simplicial objects in $\mathbf{C}$

simplicial objects explicitly:

C in $s\mathbf{C}$ consists of following data

- $C_n := C([n])$
- for $f : [n] \to [m]$ have map $f^* : C_m \to C_n$
- notation $\partial_i := d_i^*$, $\sigma_i := s_i^*$.

example:

representable simplicial sets: Δ^n in $s\mathbf{Set}$

$\Delta^n := \mathtt{Hom}_\Delta(-, [n])$

example:

- $\mathbf{C}$ - category with fibre products
- $f : X \to Y$ - morphism
- define $C(f)$ in $s\mathbf{C}$ by:
- $C(f)_n := \underbrace{X \times_Y \cdots \times_Y X}_{n+1 \text{ factors}}$
- $\phi : [m] \to [n]$ induces $\phi^* : X_n \to X_m$

– in point language: $\phi^*(x_0, \ldots, x_n) = (x_{\phi(0)}, \ldots, x_{\phi(n)})$

- $C(f)$ is called the Čech object for f

- have map $C(f) \to \underline{Y}$ ($\underline{Y}$ is constant simplicial object with value Y)
– points $(x_0, \dots, x_n) \mapsto f(x_0)$ (can also take any other entry)
- question: When is $\mathtt{colim}_{\Delta^{op}} C(f) \to Y$ an isomorphism?

define functor

$$\mathtt{sing} : \mathbf{Top} \to s\mathbf{Set} , \quad X \mapsto \mathtt{Hom}_{\mathbf{Top}}(|-|, X)$$

Definition 4.1. $\mathtt{sing}(X)$ *is called the simplicial complex of* X*.*

- $\mathtt{sing}(X)_0$ - set of points of X
- $\mathtt{sing}(X)_1$ - set of paths in X
- $\mathtt{sing}(X)_n$ - set of singular n-simplices of X

– consider γ in $\mathtt{sing}(X)_1$
– $\gamma : [0, 1] \cong \Delta^1 \to X$
— $\partial_0(\gamma) = \gamma(0)$
— $\partial_1(\gamma) = \gamma(1)$

$\sigma : \Delta^n \to X$ in $\mathtt{sing}(X)$
define support $\mathtt{supp}(\sigma) := \sigma(\Delta^n)$ - compact subset of X
- for $f : [m] \to [n]$ we have $\mathtt{supp}(f^*\sigma) \subseteq \mathtt{supp}(\sigma)$

linear simplices
V - convex subset of affine space over real vector space
- $(v_0, \dots, v_n)$ - family in V
- get singular simplex $[v_0, \dots, v_n]$ in $\mathtt{sing}(V)$

$$[v_0, \dots, v_n] : \Delta^n \to V , \quad (t_0, \dots, t_n) \mapsto t_0 v_0 + \dots + t_n v_n$$

- $\mathtt{supp}([v_0, \dots, v_n])$ is convex hull of $(v_0, \dots, v_n)$
- $\partial_m[v_0, \dots, v_n] = [v_0, \dots, \hat{v}_m, \dots v_n]$, $m = 0, \dots, n$
- $s_m[v_0, \dots, v_n] = [v_0, \dots, v_m, v_m, \dots, v_n]$, $m = 0, \dots, n$

4.2 Simplicial abelian groups and chain complexes

construct functor $C : s\mathbf{Ab} \to \mathbf{Ch}$

A in $s\mathbf{Ab}$ - simplicial abelian group

we define chain complex $C(A)$ as follows:

- $C(A)_n := A_n$
- $\partial : C_n(A) \to C_{n-1}(A)$

– $\partial := \sum_{i=0}^{n} (-1)^i \partial_i$

Lemma 4.2. $\partial \circ \partial = 0$

Proof.

consider $\partial \circ \partial : C_{n+1}(A) \to C_{n-1}(A)$

- $d_{j+1} d_i = d_i d_j$ for $i \leq j$

– how to see this:

— both maps are monotone, composition of two injective maps, hence injective

— hence coincide if they have the same image

– d_i: image does not contain i

— $d_{j+1} d_i$: image does not contain $j+1$ and i (since $i \leq j$)

– d_j: image does not contain j

— $d_i d_j$: image does not contain i and $j+1$ (since $i \leq j$)

get $\partial_i \partial_{j+1} = \partial_j \partial_i$ for $i \leq j$

$$
\begin{aligned}
\partial \circ \partial &= \sum_{i=0}^{n}(-1)^i\partial_i \circ \sum_{j=0}^{n+1}(-1)^j\partial_j \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}\partial_i\partial_j \quad \text{split sum} \\
&= \sum_{i=0}^{n}\sum_{j=i+1}^{n+1}(-1)^{i+j}\partial_i\partial_j + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_i\partial_j \quad \text{shift index first sum} \\
&= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\partial_i\partial_{j+1} + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_i\partial_j \quad \text{use relation first summand} \\
&= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\partial_j\partial_i + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_i\partial_j \quad \text{rename } i \text{ und } j \text{ second sum} \\
&= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\partial_j\partial_i + \sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i+j}\partial_j\partial_i \quad \text{switch order of second sums} \\
&= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\partial_j\partial_i + \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j}\partial_j\partial_i \quad \text{cancel} \\
&= 0
\end{aligned}
$$

□

$f : A \to B$ in $s\mathbf{Ab}$ induces map of chain complexes

$C(f) : C(A) \to C(B)$ by

$C(f)_n : A_n \to B_n$

- is chain map

get functor $C : s\mathbf{Ab} \to \mathbf{Ch}$

Definition 4.3. *We call $C(A)$ the chain complex associated to the simplicial abelian group A.*

example:

A - constant simplicial abelian group

- $A_n := A$

- all simplicial operations are `id`

- $C(A)$ has the form

– $A \overset{0}{\leftarrow} A \overset{\mathtt{id}}{\leftarrow} A \overset{0}{\leftarrow} A \overset{\mathtt{id}}{\leftarrow} A \dots$

– $H_*(C(A)) = \begin{cases} A & * = 0 \\ 0 & else \end{cases}$

$\mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Ab}$ - linearization functor (left adjoint)

- $\mathbb{Z}[-] : s\mathbf{Set} \to s\mathbf{Ab}$

for X in $s\mathbf{Set}$

- $C(\mathbb{Z}[X])$ is chain complex of simplicial set X

4.3 Singular homology

Definition 4.4. *The functor*

$$C^{\mathtt{sing}} := C(\mathbb{Z}[\mathtt{sing}(-)]) : \mathbf{Top} \to \mathbf{Ch}$$

is called the singular chain complex functor.

$$C^{\mathtt{sing}} : \mathbf{Top} \overset{\mathtt{sing}}{\to} s\mathbf{Set} \overset{\mathbb{Z}[-]}{\to} s\mathbf{Ab} \overset{C}{\to} \mathbf{Ch}$$

$C^{\mathtt{sing}}(X)$ - singular chain complex of X

- $C^{\mathtt{sing}}(X)_n$ - free group generated by singular n-simplices of X

- $A \subseteq X$ - a subspace

- inclusion $A \to X$ induces inclusion $\mathtt{sing}(A) \to \mathtt{sing}(X)$ and hence inclusion $C^{\mathtt{sing}}(A) \to C^{\mathtt{sing}}(X)$

- consider any chain complex M in $\mathbf{Ch}$

- define chain complex $C(A; M) := C(X) \otimes M$

- extend to pairs (X, A) by

$$C^{\mathtt{sing}}(X, A; M) := \frac{C(X; M)}{C(A; M)} \ .$$

get functor

$C(-, -; M) : \mathbf{Top}^2 \to \mathbf{Ch}$

Definition 4.5. *We define the singular homology functor with coefficients in M by*

$$H^{\mathtt{sing}}(-, -; M) := H(C^{\mathtt{sing}}(-, -; M)) : \mathbf{Top}^2 \to \mathbf{Ab}^{\mathbb{Z}-\mathrm{gr}} \ .$$

Example:

- $\iota_n : \Delta^n \to *$ - unique n-simplex
- $\mathtt{sing}(*)_n = \{\iota_n\}$ for all n
– $\mathtt{sing}(*)$ - constant simplicial set with value $*$
– $\mathbb{Z}[\mathtt{sing}(*)]$ - constant simplicial abelian group with value $\mathbb{Z}$
- $C^{\mathtt{sing}}(*)_n \cong \mathbb{Z}\iota_n$ for all n
- $\partial : C_n(*) \to C_{n-1}(*)$ is given by $\partial\iota_n = 1$ for n odd and $\partial\iota_n = 0$ for n even
- $C(*) : \mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{1}{\leftarrow} \mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{1}{\leftarrow}$

calculate $H_*^{\mathtt{sing}}(*; M)$

have an inclusion $i : M \to C^{\mathtt{sing}}(*; M)$, $m \mapsto \iota_0 \otimes m$

- is chain map since $\partial\iota_0 = 0$:

$$i(\partial m) \mapsto \iota_0 \otimes \partial m = \iota_0 \otimes \partial m + \partial\iota_0 \otimes m = \partial i(m)$$

Lemma 4.6. *The inclusion $i : M \to C^{\mathtt{sing}}(*; M)$ is a chain homotopy equivalence.*

Proof.

consider projection $p : C^{\mathtt{sing}}(*; M) \to M$

$$p(\iota_n \otimes m) := \begin{cases} m & n = 0 \\ 0 & else \end{cases}$$

- is chain map

$$\partial p(\iota_n \otimes m) = \begin{cases} \partial m & n = 0 \\ 0 & else \end{cases} .$$

$$p(\partial(\iota_n \otimes m)) = p(\partial\iota_n \otimes m) + (-1)^n p(\iota_n \otimes \partial m) = \begin{cases} \partial m & n = 0 \\ 0 & else \end{cases}$$

clear: $p \circ i = \mathtt{id}$

show: $i \circ p$ is chain homotopic to $\mathtt{id}$

- set $h : C^{\mathtt{sing}}(*; M) \to C^{\mathtt{sing}}(*; M)[1]$
– $h(\iota_n \otimes m) := \iota_{n+1} \otimes m.$

calculate that $\partial h(\iota_n \otimes m) + h(\partial(\iota_n \otimes m)) = \mathtt{id} - i \circ p$

- case: $n = 0$

– $-\iota_1 \otimes \partial m + \iota_1 \otimes \partial m = 0$

- case: n even, > 0

– $-\iota_{n+1} \otimes \partial m + \iota_n \otimes m + \iota_{n+1} \otimes \partial m = \iota_n \otimes m$

- case: n odd

– $\iota_n \otimes m + \iota_{n+1} \otimes \partial m - \iota_{n+1} \otimes \partial m = \iota_n \otimes m$

□

Corollary 4.7. $H^{\mathtt{sing}}(*; M) \cong H(M)$

consider pair (X, A) of top. spaces

- have natural exact sequence

$$0 \to C^{\mathtt{sing}}(A; M) \to C^{\mathtt{sing}}(X; M) \to C^{\mathtt{sing}}(X, A; M) \to 0$$

- get natural long exact sequence

$$H^{\mathtt{sing}}(A; M) \to H^{\mathtt{sing}}(X; M) \to H^{\mathtt{sing}}(X, A; M) \stackrel{\partial^{\mathtt{sing}}}{\to} H^{\mathtt{sing}}(A; M)[-1]$$

Theorem 4.8. *$(H^{\mathtt{sing}}(-, -; M), \partial^{\mathtt{sing}})$ is a homology theory with coefficients $H(M)$.*

Proof.

1. homotopy invariance
2. excision
3. additivity
4. exactess (done by definition)

□

Lemma 4.9. *$H^{\mathtt{sing}}$ is homotopy invariant.*

Proof.

X - space

must show

$i_{0,*} = i_{1,*} : H^{\mathtt{sing}}(X; M) \to H^{\mathtt{sing}}([0,1] \times X; M)$

- to this end construct chain homotopy

$$H : C^{\mathtt{sing}}(X; M) \to C^{\mathtt{sing}}([0,1] \times X; M)[1]$$

such that $\partial H + H\partial = i_{1,*} - i_{0,*}$ as follows

- consider standard simplex Δ^n

– coordinates $(\phi_1, \ldots, \phi_n)$

- form space $Z^{n+1} := [0,1] \times \Delta^n$

– have inclusions $j_0, j_1 : \Delta^n \to Z^{n+1}$, $j_\ell(x) := (\ell, x)$

— bottom and top face

– write $d^Z_m := \mathtt{id}_{[0,1]} \times d_m : Z^n \to Z^{n+1}$

— vertical boundary faces

- define singular simplices in Z^{n+1}

– $k^{n+1}_i : \Delta^{n+1} \to Z^{n+1}$, $i = 1, \ldots, n+1$ by

— $(\phi_1, \ldots, \phi_{n+1}) \mapsto (\phi_i, (\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_{n+1}))$

Δ^n inn $\mathbb{R}^{n+1}$ spanned by $e_0, \ldots, e_n$

- $Z^{n+1} \subseteq \mathbb{R} \times \mathbb{R}^{n+1}$

- k^{n+1}_i spanned by

$(0, e_{i-1}), \ldots, (0, e_n), (1, e_0), \ldots, (1, e_{i-1})$ in $\mathbb{R} \times \mathbb{R}^{n+1}$

calculate: $k^{n+1}_i \circ d_m$

- $i = 1, m = 0$

– $(\phi_1, \ldots, \phi_n) \mapsto (0, \phi_1, \ldots, \phi_n) \mapsto (0, (\phi_1, \ldots, \phi_n))$

– $k^{n+1}_1 \circ d_0 = j_0$

- $i = n+1, m = n+1$

– $(\phi_1, \ldots, \phi_n) \mapsto (\phi_1, \ldots, \phi_n, 1) \mapsto (1, (\phi_1, \ldots, \phi_n))$

– $k_{n+1}^{n+1} \circ d_{n+1} = j_1$

$i = m$ or $i = m+1$, $1 \leq i \leq n$

– $(\phi_1, \ldots, \phi_n) \mapsto (\phi_1, \ldots, \phi_i, \phi_i, \ldots \phi_n) \mapsto (\phi_i, (\phi_1, \ldots, \phi_n))$

– $k_m^{n+1} \circ d_m = k_{m+1}^{n+1} \circ d_m$

$1 \leq i \leq m-1 \leq n$

$k_i^{n+1} \circ d_m = d_{m-1}^Z \circ k_i^n$

$(\phi_1, \ldots, \phi_n) \mapsto (\phi_1, \ldots, \phi_i, \ldots, \phi_m, \phi_m, \ldots, \phi_n) \mapsto (\phi_i, (\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m, \phi_m, \ldots, \phi_n))$

$0 \leq m+2 \leq i \leq n+1$

$k_i^{n+1} \circ d_m = d_m^Z \circ k_{i-1}^n$

$(\phi_1, \ldots, \phi_n) \mapsto (\phi_1, \ldots, \phi_m, \phi_m, \ldots, \phi_{i-1}, \ldots, \phi_n) \mapsto (\phi_{i-1}, (\phi_1, \ldots, \phi_m, \phi_m, \ldots, \hat{\phi}_{i-1}, \ldots, \phi_n))$

consider

$h^{n+1} := \sum_{i=1}^{n+1} (-1)^i k_i^{n+1}$ in $C^{\mathtt{sing}}(Z^{n+1})$

$$\begin{aligned}
\partial h^{n+1} &= \sum_{m=0}^{n+1} (-1)^m \sum_{i=1}^{n+1} (-1)^i k_i^{n+1} \circ d_m \\
&= -j_0 + j_1 \\
&\quad + \sum_{m=1}^{n} [(-1)^{2m} k_m^{n+1} d_m + (-1)^{2m+1} k_{m+1}^{n+1} d_m] \\
&\quad + \sum_{m=0}^{n+1} \sum_{i=1}^{m-1} (-1)^{m+i} k_i^{n+1} \circ d_m + \sum_{m=0}^{n+1} \sum_{i=m+2}^{n+1} (-1)^{m+i} k_i^{n+1} \circ d_m \\
&= j_1 - j_0 \\
&\quad + \sum_{m=0}^{n+1} \sum_{i=1}^{m-1} (-1)^{m+i} d_{m-1}^Z \circ k_i^n + \sum_{m=0}^{n+1} \sum_{i=m+2}^{n+1} (-1)^{m+i} d_m^Z \circ k_{i-1}^n \\
&= j_1 - j_0 \\
&\quad + \sum_{m=0}^{n} \sum_{i=1}^{m} (-1)^{m+i-1} d_m^Z \circ k_i^n + \sum_{m=0}^{n} \sum_{i=m+1}^{n} (-1)^{m+i-1} d_m^Z \circ k_i^n \\
&= j_1 - j_0 - \sum_{m=0}^{n} (-1)^m \sum_{i=1}^{n} (-1)^i d_m^Z \circ k_i^n
\end{aligned}$$

– for singular simplex $\sigma : \Delta^n \to X$ set

— $\sigma^Z := (\mathtt{id}_{[0,1]} \times \sigma) : Z^{n+1} \to [0,1] \times X$

- $H(\sigma) := \sigma^Z_*(h^{n+1})$ in $C^{\mathtt{sing}}_{n+1}(X)$

– extend H linearly to coefficients, get $H : C^{\mathtt{sing}}_n(X;M) \to C^{\mathtt{sing}}_{n+1}(X;M)$

- calculate using $\sigma^Z \circ d^Z_m = (\sigma \circ d_m)^Z = (\partial_m \sigma)^Z : Z^n \to [0,1] \times X$

$$\begin{aligned}
\partial H(\sigma) &= \partial \sigma^Z_*(h^{n+1}) \\
&= \sigma^Z_*(\partial h^{n+1}) \\
&= \sigma^Z_*(j_1 - j_0 - \sum_{m=0}^{n} (-1)^m \sum_{i=1}^{n} (-1)^i d^Z_m \circ k^n_i) \\
&= i_{1,*}\sigma - i_{0,*}\sigma - \sum_{m=0}^{n} (-1)^m (\partial_m \sigma)^Z_* (\sum_{i=1}^{n} (-1)^i k^n_i) \\
&= i_{1,*}\sigma - i_{0,*}\sigma - \sum_{m=0}^{n} (-1)^m (\partial_m \sigma)^Z_* (h^n) \\
&= i_{1,*}\sigma - i_{0,*}\sigma - H(\partial \sigma)
\end{aligned}$$

hence by linear extension

$\partial \circ H + H \circ \partial = i_{1,*} - i_{0,*}$.

□

Proposition 4.10. *The functor $H^{\mathtt{sing}}(-,-;M)$ satisfies excision.*

Proof. (X,A) pair

U subset of A such that $\bar{U} \subseteq \mathrm{int}(A)$

have map of quotients

$C^{\mathtt{sing}}(X \setminus U, A \setminus U; M) \to C^{\mathtt{sing}}(X,A;M)$

- must show that this map is quasi-isomorphism

c in $C^{\mathtt{sing}}_n(X)$

- consider c as finitely supported function on $\mathtt{sing}(X)$

- $c = \sum_{\sigma \in \mathtt{sing}(X)_n} c(\sigma)\sigma$

- define support $\mathtt{supp}(c) := \bigcup_{c(\sigma) \neq 0} \mathtt{supp}(\sigma)$

call c in $C_n^{\mathtt{sing}}(X)$ small if for all σ in $\mathtt{sing}(X)_n$ with $c(\sigma) \neq 0$ we have $\mathtt{supp}(\sigma) \subseteq A$ or $\mathtt{supp}(\sigma) \subseteq X \setminus U$

assume that c is small

- $c_A := \sum_{\sigma \in \mathtt{sing}(X)_n, \mathtt{supp}(\sigma) \subseteq A} c(\sigma)\sigma$
- $c_{X \setminus U} := c - c_A$

– have $c = c_A + c_{X \setminus U}$

– $\mathtt{supp}(c_{X \setminus U}) \subseteq X \setminus U$, $\mathtt{supp}(c_A) \subseteq A$

Lemma 4.11. *For every c in $C_n^{\mathtt{sing}}(X; M)$ such that ∂c is small there exists a chain d in $C_{n+1}^{\mathtt{sing}}(X; M)$ such that $c + \partial d$ is small*

assume the lemma for the moment:

injectivity:

consider class $[[c]]$ in $H_n^{\mathtt{sing}}(X \setminus U, A \setminus U; M)$

- represented by cycle $[c]$ in $C_n^{\mathtt{sing}}(X \setminus U, A \setminus U; M)$
- c in $C_n^{\mathtt{sing}}(X \setminus U; M)$

– $\partial[c] = 0$ means $\partial c \in C_{n-1}^{\mathtt{sing}}(A \setminus U; M)$

- assume: image of $[[c]]$ in $H_n^{\mathtt{sing}}(X, A; M)$ vanishes

– there is u in $C_n(X; M)$ such that $c + \partial u \in C_n(A; M)$

– $\partial u = -c + d$ with $\mathtt{supp}(c) \subseteq X \setminus U$ and $\mathtt{supp}(d) \subseteq A$

– ∂u is small

– let v be chosen by the lemma such that $u + \partial v =: u'$ is small

– decompose $u' = u'_A + u'_{X \setminus U}$

– $c + \partial u = c + \partial(u + \partial v) = c + \partial u' = c + \partial u'_A + \partial u'_{X \setminus U}$

— conclude $c + \partial u'_{X \setminus U} \in C_n^{\mathtt{sing}}(A \setminus U)$.

– hence $[[c]] = 0$ in $H_n^{\mathtt{sing}}(X \setminus U, A \setminus U; M)$

surjectivity:

$[[c]]$ in $H_n^{\mathtt{sing}}(X, A; M)$

- $[c]$ cycle in $C_n^{\mathtt{sing}}(X, A; M)$

– c in $C_n^{\mathtt{sing}}(X; M)$

- $\partial c \in C^{\mathtt{sing}}_{n-1}(A;M)$

– ∂c is small

– chose d such that $c+\partial d$ is small

– $c+\partial d = c' = c'_{X\setminus U} + c'_A$

– $c'_{X\setminus U}$ in $C^{\mathtt{sing}}_n(X\setminus U;M)$

– $\partial c'_{X\setminus U} = \partial c - \partial c'_A$ is supported in $A\setminus U$

– $\partial [c'_{X\setminus U}] = 0$

- $[[c]] = [[c]+\partial[d]] = [[c']] = [[c'_{X\setminus U}]]$ in $H^{\mathtt{sing}}(X,A;M)$

$[[c]]$ is in the image

□

prepare proof of Lemma

V convex in real affine space

- singular simplices of the form $[v_0,\dots,v_n]$ are called linear

– are preserved by simplicial operations

- $L\mathtt{sing}(V)$ - subsimplicial set of $\mathtt{sing}(V)$ of linear simplices

- $C^{L\mathtt{sing}}(V) := C(\mathbb{Z}[L\mathtt{sing}(V)])$ - subcomplex of $C^{L\mathtt{sing}}(V)$ of chains of linear simplices

- for affine map $f: V\to V'$ get chain map $f_*: C^{L\mathtt{sing}}(V)\to C^{L\mathtt{sing}}(V')$

$\sigma = [v_0,\dots,v_n]$ singular simplex

Definition 4.12. *$b_\sigma := \frac{1}{n+1}v_0 + \dots + \frac{1}{n+1}v_n$ is called barycenter of σ*

fix b in V

- get map $b: L\mathtt{sing}(V)_n \to L\mathtt{sing}(V)_{n+1}$

– $[v_0,\dots,v_n]\mapsto [b,v_0,\dots,v_n]$

– extend linearly to $b: C^{L\mathtt{sing}}_n(X)\to LC^{L\mathtt{sing}}_{n+1}(X)$

– calculate

$$\begin{aligned}
\partial b([v_0,\ldots,v_n]) &= \partial[b,v_0,\ldots,v_n] \\
&= [v_0,\ldots,v_n] - \sum_{m=0}^{n}(-1)^m[b,v_0,\ldots,\hat{v}_m,\ldots,v_n] \\
&= [v_0,\ldots,v_n] - b(\partial[v_0,\ldots,v_n])
\end{aligned}$$

shortly: $\partial b + b\partial = \mathtt{id}$

define subdivision operator $S : C_n^{L\mathtt{sing}}(V) \to C_n^{L\mathtt{sing}}(V)$ by induction by n

set $C_{-1}^{L\mathtt{sing}}(V) := \mathbb{Z}$ with generator $[]$

start: $n = -1$

-$S([]) := []$, $S = \mathtt{id}$ on $C_{-1}^{L\mathtt{sing}}(X)$

step n:

- assume: S defined on $C_m^{L\mathtt{sing}}(V)$ for $m < n$
- σ in $L\mathtt{sing}(V)_n$
- $S(\sigma) := b_\sigma(S(\partial\sigma))$
- linear extension

check: S is chain map

induction by n

start: $n = -1$

- $\partial S([]) = 0 = S(\partial[])$

step: assume for $m < n$:

$$\begin{aligned}
\partial S(\sigma) &= \partial b_\sigma(S(\partial\sigma)) \quad \partial b = \mathtt{id} - b\partial \\
&= S(\partial\sigma) - b_\sigma(\partial S(\partial\sigma)) \quad \text{induction hypothesis} \\
&= S(\partial\sigma) - b_\sigma(S(\partial\partial\sigma)) \quad \partial\partial = 0 \\
&= S(\partial\sigma)
\end{aligned}$$

note:

- $\mathtt{supp}(S(c)) \subseteq \mathtt{supp}(c)$

- for affine map $f : V \to V'$

– $S \circ f_* = f_* \circ S$

define chain homotopy $T : C_n^{L\mathtt{sing}}(V) \to C_n^{L\mathtt{sing}}(V)$

start:

$n = -1$

$T([]) := 0$

step:

- assume: T defined for $m < n$
- σ in $L\mathtt{sing}(V)_n$
- $T(\sigma) := b_\sigma(\sigma - T(\partial\sigma))$

calculate (by induction):

$\partial T + T\partial = \mathtt{id} - S$

start: $n = -1$

$(\partial T + T\partial)([]) = 0 = (\mathtt{id} - S)([])$

step: assume for $m < n$

- σ in $L\mathtt{sing}(V)_n$

$$
\begin{aligned}
\partial T(\sigma) &= \partial b_\sigma(\sigma - T(\partial\sigma)) \quad \partial b = \mathtt{id} - b\partial \\
&= \sigma - T(\partial\sigma) - b_\sigma(\partial(\sigma - T(\partial\sigma))) \\
&= \sigma - T(\partial\sigma) - b_\sigma(\partial\sigma) + b_\sigma(\partial T(\partial\sigma)) \quad \text{induction assumption} \\
&= \sigma - T(\partial\sigma) - b_\sigma(\partial\sigma) + b_\sigma(\partial\sigma - S(\partial\sigma) - T(\partial\partial\sigma)) \quad S(\sigma) = b_\sigma(S(\partial\sigma)) \\
&= \sigma - T(\partial\sigma) - S(\sigma)
\end{aligned}
$$

note:

- $\mathtt{supp}(T(c)) \subseteq \mathtt{supp}(c)$.
- for affine map $V \to V'$
- $T \circ f_* = f_* \circ T$

consider $\mathbb{R}^\infty$ with basis $e_0, e_1, \ldots$

- identify space Δ^n with $\mathtt{supp}([e_0, \ldots, e_n])$

– is convex subset

– $[e_0, \ldots, e_n] \in L\mathtt{sing}(\Delta^n)_n$

- $S([e_0, \ldots, e_n]) \in C_n^{L\mathtt{sing}}(\Delta^n)$

X - space

- σ in $\mathtt{sing}(X)_n$

- $\sigma = \sigma_*([e_0, \ldots, e_n])$ in $C_n^{\mathtt{sing}}(X)$

define subdivision operator $S : C_n^{\mathtt{sing}}(X) \to C_n^{\mathtt{sing}}(X)$ by

- $S(\sigma) := \sigma_*(S([e_0, \ldots, e_n]))$

- linear extension

- check: S is chain map

$$\begin{aligned}
\partial S(\sigma) &= \partial \sigma_*(S([e_0, \ldots, e_n])) \\
&= \sigma_*(\partial S([e_0, \ldots, e_n])) \\
&= \sigma_*(S(\partial [e_0, \ldots, e_n])) \\
&= \sum_{m=0}^{n} (-1)^m \sigma_*(S([e_0, \ldots, \hat{e}_m, \ldots, e_n])) \\
&\overset{!}{=} \sum_{m=0}^{n} (-1)^m (\partial_m \sigma)_*(S([e_0, \ldots, e_{n-1}])) \\
&= \sum_{m=0}^{n} (-1)^m S(\partial_m \sigma) \\
&= S(\partial \sigma)
\end{aligned}$$

for marked equality:

- use affine map $d_m : \Delta^{n-1} \to \Delta^n$

- $d_{m,*}[e_0, \ldots, e_{n-1}] = [e_0, \ldots, \hat{e}_m, \ldots, e_n]$

$$\begin{aligned}
\sigma_*(S([e_0, \ldots, \hat{e}_m, \ldots, e_n])) &= \sigma_*(S(d_{m,*}[e_0, \ldots, e_{n-1}])) \\
&= \sigma_*(d_{m,*}(S([e_0, \ldots, e_{n-1}]))) \\
&= (\partial_m \sigma)_*(S([e_0, \ldots, e_{n-1}]))
\end{aligned}$$

note: if c is small, then so is $S(\sigma)$

define homotopy $T: C_n^{\mathtt{sing}}(X) \to C_{n+1}^{\mathtt{sing}}(X)$ by

- $T(\sigma) := \sigma_*(T([e_0, \dots, e_n]))$

– linear extension

– check: $\partial T + T\partial = \mathtt{id} - S$

$$\begin{aligned}
\partial T(\sigma) &= \partial\sigma_*(T([e_0, \dots, e_n])) \\
&= \sigma_*(\partial T([e_0, \dots, e_n])) \\
&= \sigma_*(T(\partial[e_0, \dots, e_n]) + [e_0, \dots, e_n] - S([e_0, \dots, e_n])) \\
&= \sum_{m=0}^{n} (-1)^m \sigma_*(T([e_0, \dots, \hat{e}_m, \dots, e_n])) + \sigma - S(\sigma) \\
&= \sum_{m=0}^{n} (-1)^m (\partial_m\sigma)_*(T([e_0, \dots, e_{n-1}])) + \sigma - S(\sigma) \\
&= T(\partial\sigma) + \sigma - S(\sigma)
\end{aligned}$$

note: if c is small, then so is $T(c)$

Proof. (of Lemma)

use euclidean metric on $\mathbb{R}^\infty$

for subset A of $\mathbb{R}^n$ define diameter

$$\mathrm{diam}(A) := \sup_{x,y \in A} d(x, y)$$

- for $[v_0, \dots, v_n]$ in $L\mathtt{sing}(A)_n$

- $\mathrm{diam}(\mathtt{supp}([v_0, \dots, v_n])) = \max_{0 \le i,j \le n} \|v_i - v_j\|$

– indeed for any v in $\mathtt{supp}([v_0, \dots, v_n])$

$$\|v - \sum_{i=0}^{n} t_i v_i\| = \|\sum_{i=0}^{n} t_i(v - v_i)\| \le \sum_{i=0}^{n} t_i\|v - v_i\| \le \sum_{i=0}^{n} t_i \max_i \|v - v_i\| = \max_i \|v - v_i\| \ .$$

observe:

$\mathrm{diam}(\Delta^n) = \max_{0 \le i,j \le n} \|e_i - e_j\| = \sqrt{2}$

- $\sigma \in L\mathtt{sing}(A)_n$

- $\mathrm{Lip}(\sigma) \le \frac{\mathrm{diam}(\sigma)}{\mathrm{diam}(\Delta^n)} = \frac{\mathrm{diam}(\sigma)}{\sqrt{2}}$

for chain c in $C_n^{\texttt{sing}}(A)$ define

$$\operatorname{diam}(c) = \max_{\sigma \in \texttt{sing}(A)_n, c(\sigma) \neq 0} \operatorname{diam}(\texttt{supp}(\sigma))$$

claim: $\operatorname{diam}(S(c)) \leq \frac{n}{n+1}\operatorname{diam}(c)$

verify by induction on n:

- recall $S(\sigma) = b_\sigma(S(\partial\sigma))$
- $\operatorname{diam}(S(\partial\sigma)) \leq \frac{n-1}{n}\operatorname{diam}(\partial\sigma) \leq \frac{n-1}{n}\operatorname{diam}(\sigma)$
- $\lambda = [w_0, \dots, w_n]$ - singular simplex contributing to $S(\partial\sigma)$
- know by induction hypothesis: $\operatorname{diam}(\lambda) \leq \frac{n-1}{n}\operatorname{diam}(\sigma)$
- consider $\operatorname{diam}(b_\sigma(\lambda)) = \operatorname{diam}([b_\sigma, w_0, \dots, w_n])$

– $\|w_i - w_j\| \leq \frac{n-1}{n}\operatorname{diam}(\sigma) \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$

– $\|b_\sigma - w_i\| \leq \max_k \|b_\sigma - v_k\|$

— $b_k := \frac{1}{n}\sum_{i=0, i\neq k}^{n} v_i$

— $b = \frac{n}{n+1}b_k + \frac{1}{n+1}v_k$

— $b - v_k = \frac{n}{n+1}(b_i - v_k)$

— $\|b - v_k\| = \frac{n}{n+1}\|b_i - v_k\| \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$

– conclude $\|b_\sigma - w_i\| \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$

$\sigma : \Delta^n \to X$

- $\sigma^{-1}(\bar{U})$, $\sigma^{-1}(X \setminus \operatorname{int}(A)$ are closed and disjoint
- $\operatorname{sep}(\sigma) := \operatorname{dist}(\sigma^{-1}(\bar{U}), \sigma^{-1}(X \setminus \operatorname{int}(A)) > 0$ by compactness of Δ
- define for chain: $\operatorname{sep}(c) = \inf_{\sigma \in \texttt{sing}(X), c(\sigma)\neq 0} \operatorname{sep}(\sigma)$
- note: $\operatorname{sep}(c) > 0$

if $\operatorname{sep}(\sigma) > \sqrt{2}$, then σ is small, i.e. $\texttt{supp}(\sigma) \subseteq A$ or $\texttt{supp}(\sigma) \subseteq X \setminus U$.

- argue by contradiction
- assume $\texttt{supp}(\sigma) \not\subseteq A$ and $\texttt{supp}(\sigma) \not\subseteq X \setminus U$
- take t in Δ^n such that $\sigma(t) \in X \setminus A \subseteq X \setminus \operatorname{int}(A)$
- take s in Δ^n such that $\sigma(s) \in U \subseteq \bar{U}$
- then $\operatorname{sep}(\sigma) \leq \|s - t\| \leq \sqrt{2}$

$\mathrm{sep}(S(\sigma)) \geq \frac{n+1}{n}\mathrm{sep}(\sigma)$

- $S(\sigma) = \sigma_*(S([e_0, \ldots, e_n]))$
- λ a simplex contributing to $S([e_0, \ldots, e_n])$
- $\mathrm{diam}(\lambda) \leq \frac{n}{n+1}\sqrt{2}$
- $\mathrm{Lip}(\lambda) \leq \frac{\mathrm{diam}(\lambda)}{\sqrt{2}} \leq \frac{n}{n+1}$
- $\mathrm{sep}(\lambda) \geq \frac{\mathrm{sep}(\sigma)}{\mathrm{Lip}(\lambda)} \geq \frac{n+1}{n}\mathrm{sep}(\sigma)$
- conclude $\mathrm{sep}(S(c)) \geq \frac{n+1}{n}\mathrm{sep}(c)$

consider chain c in $C_n^{\mathtt{sing}}(X)$ such that ∂c is small

define

$$
\begin{aligned}
c - \partial T(c) &= T(\partial c) + S(c) \\
c - \partial T(c) - \partial T(S(c)) &= T(\partial c) + S(c) + T(\partial S(c)) - S(c) + S^2(c) \\
&= T(\partial c) + T(S(\partial c)) + S^2(c) \\
c - \partial \sum_{i=0}^{k-1} T(S^i(c)) &= \sum_{i=0}^{k-1} T(S^i(\partial c)) + S^k(c)
\end{aligned}
$$

- $\sum_{i=0}^{k-1} T(S^i(\partial c))$ is small
- $\mathrm{sep}(S^k(c)) \geq (\frac{n+1}{n})^k \mathrm{sep}(c)$
- choose k so large that $(\frac{n+1}{n})^k \mathrm{sep}(c) \geq \sqrt{2}$
- then $S^k(c)$ is also small
- in this case with $d := \sum_{i=0}^{k-1} T(S^i(c))$

– $c + \partial d$ is small

□

Proposition 4.13. *$H^{\mathtt{sing}}(-; M)$ satisfies the additivity axiom.*

Proof.

$(X_i, A_i)_{i\in I}$ family in $\mathbf{Top}^2$

$(X, A) := \coprod_{i\in I}(X_i, A_i)$

$\mathtt{sing}(X) \cong \coprod_i \mathtt{sing}(X_i)$

- since Δ^n is connected for all n

$\mathbb{Z}[-]$ preserves coproducts (left adjoint)

- $\mathbb{Z}[\mathtt{sing}(X)] \cong \bigoplus_{i\in I} \mathbb{Z}[\mathtt{sing}(X_i)]$
- $C^{\mathtt{sing}}(X) \cong \bigoplus_{i\in I} C^{\mathtt{sing}}(X_i)$

similar for A

$-\otimes M$ and quotients commute with sums

- $C^{\mathtt{sing}}(X, A; M) \cong \bigoplus_{i\in I} C^{\mathtt{sing}}(X_i, A_i; M)$
- finally: H distributes over sums (exercise)

□

4.4 Additional properties of $H^{\mathtt{sing}}$

(X, A) - pair of spaces

- $\bar{A} \subseteq U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq X$ increasing family of open subsets
- $X = \bigcup_{n\in\mathbb{N}} U_n$

Lemma 4.14. *The natural map is an isomorphism*

$$\operatorname*{\mathtt{colim}}_{n\in\mathbb{N}} H^{\mathtt{sing}}(U_n, A; M) \xrightarrow{\cong} H^{\mathtt{sing}}(X, A; M) \ .$$

Proof.

preparations:

K compact subset of X

- $(U_n \cap K)_{n\in\mathbb{N}}$ open covering of K
- there exists n such that $K \subseteq U_n$

c in $C(X, M)$

- $\mathtt{supp}(c)$ is compact

injectivity:

$[[c]]$ in $H^{\mathtt{sing}}(U_n, A; M)$ represents zero in $H^{\mathtt{sing}}(X, A; M)$

- there exists d in $C^{\mathtt{sing}}(X, M)$ with $c + \partial d \in C^{\mathtt{sing}}(A; M)$
- there exists $k \geq n$ in $\mathbb{N}$ such that $d \in C^{\mathtt{sing}}(U_k, M)$
- then image of $[[c]]$ in $H^{\mathtt{sing}}(U_k, A; M)$ zero

- hence image of $[[c]]$ in colimit zero

surjectivity

- $[[c]]$ in $H^{\mathtt{sing}}(X, A; M)$
- there exists n in $\mathbb{N}$ such that d in $C^{\mathtt{sing}}(U_n; M)$
- hence $[[c]]$ exists already in $H^{\mathtt{sing}}(U_n, A; M)$

□

X - any space

Lemma 4.15. $H_0^{\mathtt{sing}}(X) \cong \mathbb{Z}[\pi_0^{path}(X)]$

Proof.

surjective map $\mathtt{sing}(X)_0 \to X \to \pi_0^{path}(X)$ induces

surjective map $C_0^{\mathtt{sing}}(X) \to \mathbb{Z}[\pi_0^{path}(X)]$

- $c \mapsto \sum_{x \in X} c(x)[x]^{path} = \sum_{W \in \pi_0(X)} (\sum_{x \in W} c(x)) W$

claim: $C_1^{\mathtt{sing}}(X) \xrightarrow{\partial} C_0^{\mathtt{sing}}(X) \to \mathbb{Z}[\pi_0^{path}(X)]$ vanishes

- $\gamma \in \mathtt{sing}(X)_1$
- $\gamma \mapsto [\gamma(1)]^{path} - [\gamma(0)]^{path} = 0$

get induced surjective map

$H_0(X) \to \mathbb{Z}[\pi_0^{path}(X)]$

injective:

- for every W in $\pi_0(X)$ fix base point b_W in X
- assume: c in $C_0^{\mathtt{sing}}(X)$, $c \mapsto 0$
- $\sum_{x \in W} c(x) = 0$ for all W in $\pi_0^{path}(X)$
- chose for every x in X path γ_x from $b_{[x]^{path}}$ to x
- define $d := \sum_{x \in X} c(x)\gamma_x$ in $C_1^{\mathtt{sing}}(X)$,

$$
\begin{aligned}
\partial d &= \sum_{x\in X} c(x)x - \sum_{x\in X} c(x)b_{[x]^{path}} \\
&= c - \sum_{W\in\pi_0^{path}(X)} \sum_{x\in W} c(x)b_{[x^{path}]} \\
&= c - \sum_{W\in\pi_0^{path}(X)} \left(\sum_{x\in W} c(x)\right) b_W \\
&= c
\end{aligned}
$$

hence $[c]=0$ in $H_0^{\mathtt{sing}}(X)$ □

4.5 Jordan curve theorem and other applications

recall: for non-empty X we have $\tilde{H}(X)\oplus H(*)\cong H(X)$

- $\tilde{H}(X) := \mathtt{ker}(H(X)\to H(*))$
- use retraction $*\to X\to *$ so see splitting
- what about $X=\emptyset$?

definition is not good homotopically

more precise definition:

- define $\tilde{H}^{\mathtt{sing}}(X)$ as homology of homotopy fibre of $C^{\mathtt{sing}}(X)\to C^{\mathtt{sing}}(*)$
- homotopy fibre of a surjective map between chain complexes is kernel
- in general it is represented by $\mathtt{Cone}(C^{\mathtt{sing}}(X)\to C^{\mathtt{sing}}(*))[1]$
- justification in homological algebra

for empty set: $C^{\mathtt{sing}}(\emptyset)=0$

$\mathtt{Cone}(0\to C^{\mathtt{sing}}(*))\simeq C^{\mathtt{sing}}(*)$

- so $\tilde{H}^{\mathtt{sing}}(\emptyset)\cong H(C^{\mathtt{sing}}(*)[1])\cong \mathbb{Z}[1]$
- $\tilde{H}_*^{\mathtt{sing}}(\emptyset)\cong \begin{cases}\mathbb{Z} & *=-1\\ 0 & else\end{cases}$

look at Mayer-Vietoris X decomposed into non-empty subsets U and V such that $U\cap V=\emptyset$

$0\to \tilde{H}_0^{\mathtt{sing}}(U)\oplus\tilde{H}_0^{\mathtt{sing}}(V)\to\tilde{H}_0^{\mathtt{sing}}(X)\to\mathbb{Z}\to 0$ is exact

in the following consider $H=H^{\mathtt{sing}}$

Proposition 4.16.

1. *For an embedding* $h : D^k \to S^n$ *we have* $\tilde{H}(S^n - h(D^k)) \cong 0$.
2. *For an embedding* $h : S^k \to S^n$ *with* $k < n$ *we have* $\tilde{H}(S^n - h(S^k)) \cong \mathbb{Z}[-n+k+1]$.

Proof.

(1)

induction by k

start: $k = 0$

- $S^n \setminus h(D^0) \cong \mathbb{R}^n$
- $\tilde{H}(\mathbb{R}^n) \cong 0$

step $k-1 \Rightarrow k$

by contradiction:

- assume: $\tilde{H}(S^n \setminus h(I^k)) \ncong 0$
- fix non-zero class ϕ
- identify $D^k \cong I^k$ with $I = [0,1]$
- set $A := S^n \setminus h(I^{k-1} \times [0,1/2])$ and $B := S^n \setminus h(I^{k-1} \times [1/2,1])$

– $A \cap B = S^n \setminus h(I^k)$

– $A \cup B = S^n \setminus h(I^{k-1} \times \{1/2\})$ (since h is injective)

Mayer-Vietoris sequence and induction hypothesis

$\tilde{H}(S^n \setminus h(I^k)) \xrightarrow{\cong} \tilde{H}(A) \oplus \tilde{H}(B)$

- then image of ϕ in one of $\tilde{H}(A)$ and $\tilde{H}(B)$ is non-trivial

– after rescaling can e.g. identify $A \cong S^n \setminus h'(I^k)$ for new map h'

repeat the argument

- get nested sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \dots$ such that image of ϕ in $\tilde{H}(S^n \setminus h(I^{k-1} \times I_\ell))$ is non-trivial
- $S^n \setminus h(I^{k-1} \times I_\ell)$ is open
- $\mathtt{colim}_\ell \, S^n \setminus h(I^{k-1} \times I_\ell) = S^n \setminus h(I^{k-1})$

– $\tilde{H}(S^n \setminus h(I^{k-1})) \cong \mathtt{colim}_\ell \, \tilde{H}(S^n \setminus h(I^{k-1} \times I_\ell))$

– here important that $\tilde{H} = \tilde{H}^{\mathtt{sing}}$

- the class ϕ represents a non-trivial element in the colimit

– get non-trivial class in $\tilde{H}(S^n \setminus h(I^{k-1}))$ - contradiction to induction hypothesis

(2)

induction by k

start $k = 0$

$S^n \setminus h(S^0) \cong S^{n-1} \times \mathbb{R} \sim S^{n-1}$

assertion is true by calculation of homology of sphere

step: $k - 1 \Rightarrow k$

write $S^k \cong D^k_+ \cup_{S^{k-1}} D^k_-$ (upper and lower hemisphere)

$A := S^n \setminus h(D^k_+)$ and $B := S^n \setminus h(D^k_-)$

- Mayer-Vietoris and assertion 1

$H(S^n \setminus h(S^{k-1})) \cong \tilde{H}(S^n \setminus h(S^k))[-1]$

- using induction hypothesis

$\mathbb{Z}[-n + (k-1) + 1] \cong H(S^n \setminus h(S^k))[-1]$

$\mathbb{Z}[-n + k - 1 + 1 + 1] = \mathbb{Z}[-n + k + 1] = H(S^n \setminus h(S^k))$ □

Corollary 4.17.

1. *Every embedding $S^n \to S^n$ is surjective.*
2. *There is no continuous embedding $S^n \to \mathbb{R}^n$.*
3. *There is no continuous injection $\mathbb{R}^m \to \mathbb{R}^n$ for $m > n$*

Proof.

(1)

consider embedding $h : S^n \to S^n$

- MV from proof in case $k = n$ ends with

$\tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(S^n \setminus h(S^{n-1}) \to \tilde{H}_{-1}(S^n \setminus h(S^n)) \to 0$

- by Prop. 4.16.(1) have $\tilde{H}_0(A) \oplus \tilde{H}_0(B) \cong 0$

- by Prop. 4.16.(2) have $\tilde{H}_0(S^n \setminus h(S^{n-1}) \cong \mathbb{Z}$

- hence $\tilde{H}_{-1}(S^n \setminus h(S^n)) \cong \mathbb{Z}$

-hence $S^n \setminus h(S^n) \cong \emptyset$

- hence h surjective

(2)

assume $f : S^n \to \mathbb{R}^n$ embedding

- get non-surjective embedding $h : S^n \to \mathbb{R}^n \to S^n$

- contradiction

(3)

assume $f : \mathbb{R}^m \to \mathbb{R}^n$ continuous embedding, $m > n$

- get continuous non-surjective embedding $S^n \to \mathbb{R}^m \to \mathbb{R}^n \to S^n$

- contradiction

□

Corollary 4.18 (Jordan curve theorem)**.** *If $h : S^{n-1} \to S^n$ is an embedding, then* $|\pi_0^{path}(S^n \setminus h(S^{n-1}))| = 2$

$$\begin{aligned} |\pi_0^{path}(S^n \setminus h(S^{n-1}))| &= \mathtt{rk} H_0^{\mathtt{sing}}(S^n \setminus h(S^{n-1})) \\ &= \mathtt{rk} \tilde{H}_0^{\mathtt{sing}}(S^n \setminus h(S^{n-1})) + 1 \\ &= \mathtt{rk}(\mathbb{Z}) + 1 \\ &= 2 \end{aligned}$$

Proposition 4.19 (Invariance of the domain)**.** *Let U be open in $\mathbb{R}^n$ and $h : U \to \mathbb{R}^n$ be an embedding. Then $h(U)$ is open in $\mathbb{R}^n$.*

Proof.

view $\mathbb{R}^n$ as subspace of S^n

- enough to show that $h(U)$ is open in S^n

- consider x in U

- D a disc arround x in U

- enough to show that $h(D \setminus \partial D)$ is open in S^n

- $S^n \setminus h(\partial D)$ has two path components

– given by $h(D \setminus \partial D)$ and $S^n \setminus h(D)$

— the two subsets are disjoint

— the first is a path component since $D \setminus \partial D$ is connected

— the second is a path component since there are only two

$S^n \setminus h(\partial D)$ is open

- path components are components

- $S^n \setminus h(\partial D)$ has finitely many components, those are then open

- $h(D \setminus \partial D)$ is open

□

Corollary 4.20. *Let M be a compact and N be a connected n-dimensional manifold. Then every embedding $h : M \to N$ is a homeomorphism.*

Proof.

$h(M)$ is closed in N by compactness of M and since N is Hausdorff

- since N is connected its enough to show that $h(M)$ is also open

– this follows from Invariance of Domain

□

Proposition 4.21. *If A is a finite-dimensional commutative unital division algebra over $\mathbb{R}$, then $A \cong \mathbb{R}$ or $A \cong \mathbb{C}$.*

Proof.

can assume $A = \mathbb{R}^n$

- with product $\mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}^n$

define $f : S^{n-1} \to S^{n-1}$ by $f(x) := \frac{x^2}{\|x^2\|}$

- well defined since $x \neq 0$ implies $x^2 \neq 0$ (division algebra has inverses)

- observe: $f(-x) = f(x)$

- get induced map $\bar{f} : \mathbb{RP}^{n-1} \to S^{n-1}$

- claim: $\bar{f}$ is injective

– assume $\bar{f}([x]) = \bar{f}([y])$ for x, y in S^{n-1}

– $\frac{x^2}{\|x^2\|} = \frac{x^2}{\|y^2\|}$

– $x^2 = \alpha y^2$ for $\alpha = \frac{\|x^2\|}{\|y^2\|} > 0$

– $(x - \alpha y)(x + \alpha y) = 0$

– $x = \pm \alpha y$

– since x and y are normalized and $\alpha > 0$ $x = \pm y$

– $[x] = [y]$

– finish of proof of claim

$\bar{f}$ injective map between compact Hausdorff spaces

- $\bar{f}$ is embedding

- $\bar{f}$ is surjective (if not $n = 1$, since S^0 is not connected)

- $\mathbb{PR}^{n-1} \cong S^{n-1}$ implies $n \in 2$

conclude: $\dim_{\mathbb{R}}(A) \in \{1, 2\}$

case $\dim_{\mathbb{R}}(A) = 1$: then $A \cong \mathbb{R}$

case $\dim_{\mathbb{R}}(A) = 2$: then $A \cong \mathbb{C}$ (exercise)

□

Remark: $\mathbb{H}$ is a 4-dimensional division algebra over $\mathbb{R}$, but not commutative

4.6 Universal coefficient theorem

C in **Ch**

A in **Ab**

- what is the relation between $H_*(C) \otimes A$ and $H_*(C \otimes A)$

Proposition 4.22. *Assume that C consists of free abelian groups. Then for every n in $\mathbb{N}$ we have a split short exact sequence*

$$0 \to H_n(C) \otimes A \to H_n(C \otimes A) \to \mathtt{Tor}(H_{n-1}(C), A) \to 0$$

Proof.

subgroups of free abelian groups are free abelian

- $0 \to B_n \to Z_n \to H_n(C) \to 0$

- is a free resolution of $H_n(X)$

- apply $- \otimes A$

$$0 \to \mathtt{Tor}(H_n(C), A) \to B_n \otimes A \xrightarrow{i_n} C_n \otimes A \to H_n(C) \otimes A \to 0$$

have exact sequence of vertical chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial} & B_{n-2} & \longrightarrow & 0
\end{array}$$

since B_n are free - horizontal sequences split

- tensoring with A gives exact sequence of vertical chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n \otimes A & \longrightarrow & C_n \otimes A & \xrightarrow{\partial} & B_{n-1} \otimes A & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_{n-1} \otimes A & \longrightarrow & C_{n-1} \otimes A & \xrightarrow{\partial} & B_{n-2} \otimes A & \longrightarrow & 0
\end{array}$$

Snake Lemma

$$\overset{i_n}{\dots}\to Z_n \otimes A \to H_n(C \otimes A) \to B_{n-1} \otimes A \overset{i_{n-1}}{\to} Z_{n-1} \otimes A \to H_{n-1}(C \otimes A) \to B_{n-2} \otimes A \to \dots$$

i_{n-1} - inclusion

$$0 \to \operatorname{coker}(i_n) \to H_n(C \otimes A) \to \texttt{ker}(i_n) \to 0$$

read off

$$0 \to H_n(C) \otimes A \to H_n(C \otimes A) \to \texttt{Tor}(H_{n-1}(C), A) \to 0$$

get split map $s : C_n \to Z_n$

- get map $\bar{s} : C_n \to H_n(C)$

- get chain map $\bar{s} : C \to H(C)$

– check compatibility with differential

– $0 = \partial \bar{s}(c)$

– $\bar{s}(\partial c) = 0$ (since $\partial c \in Z_{n-1}$ and $s(\partial c) = \partial c$.

get map

$C \otimes A \to H(C) \otimes A$

finally

$H(C \otimes A) \to H(C) \otimes A$

this map splits the sequences □

calculate tor groups

if A is free, then $\mathtt{Tor}(H, A) = 0$ for every abelian group H

- A is its own free resolution

p, q prime

$$\mathtt{Tor}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{Z}/q^m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p^{m \wedge n}\mathbb{Z} & p = q \\ 0 & else \end{cases}$$

- $0 \to \mathbb{Z} \xrightarrow{p^n} \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ is free resolution
- apply $- \otimes \mathbb{Z}/q^m\mathbb{Z}$
- study kernel of $\mathbb{Z}/q^m\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}/q^m$
- case: $p \neq q$

– p^n is iso

- case: $p = q$

– $\mathbb{Z}/p^{m \wedge n}\mathbb{Z}$

$\mathtt{Tor}(H, A \oplus A') \cong \mathtt{Tor}(H, A) \oplus \mathtt{Tor}(H, A')$

$\mathtt{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ with $d = gcd(n, m)$

X, A be a pair of topological spaces

G - abelian group

Corollary 4.23. *For every n in $\mathbb{N}$ we have a split exact sequence*

$$0 \to H_n^{\mathtt{sing}}(X, A) \otimes G \to H_n^{\mathtt{sing}}(X, A; G) \to \mathtt{Tor}(H_{n-1}^{\mathtt{sing}}(X, A), G) \to 0 \ .$$

Proof. $C(X, A)$ consists of free abelian groups □

example:

$$\text{k odd } H_n^{\texttt{sing}}(\mathbb{RP}^k) \cong \begin{cases} \mathbb{Z} & n = 0, k \\ 0 & n \text{ even or } n > k \\ \mathbb{Z}/2\mathbb{Z} & n \text{ odd, } n = 1, 3, \ldots, k-2 \end{cases}$$

$$H_n^{\texttt{sing}}(\mathbb{RP}^k; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 0 \ldots k \\ 0 & else \end{cases}$$

case: $n = 0$ clear

case: n even, $1 < n \leq k$

- $H_n^{\texttt{sing}}(\mathbb{RP}^k; \mathbb{Z}/2\mathbb{Z}) \cong \texttt{Tor}(H_{n-1}^{\texttt{sing}}(\mathbb{RP}^k), \mathbb{Z}/2\mathbb{Z}) \cong \texttt{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

case: n odd

- $H_n^{\texttt{sing}}(\mathbb{RP}^k; \mathbb{Z}/2\mathbb{Z}) \cong H_n^{\texttt{sing}}(\mathbb{RP}^k) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

4.7 Cohomology

fix abelian group G

- have functor: $\texttt{Hom}(-, G) : \mathbf{Ch}^{\text{op}} \to \mathbf{Ch}$

– degree convention: $\texttt{Hom}(-, G)_n := \texttt{Hom}(C_{-n}, G)$

– differential: $\partial : \texttt{Hom}(-, G)_n \to \texttt{Hom}(-, G)_{n-1}$

– $\partial\phi := (-1)^{\deg(c)}\phi \circ \partial$

— if $\deg(\phi) = n$ then $\phi : C_{-n} \to G$, $\partial\phi : C_{-n+1} \to G$

— $-n + 1 = -(n-1)$, hence $\deg(\phi) = n - 1$

often use convention $C^n := C_{-n}$

- then write d instead of partial

- $d : C^n \to C^{n+1}$ (increases degree)

- this is called the cohomological grading

example

- de Rham complex

- M manifold

- $d : \Omega^n(M) \to \Omega^{n+1}(M)$ - de Rham differential

$\texttt{Hom}(-, G)$ is not exact

$0 \to A \to B \to C \to 0$ short exact sequence in **Ab**

apply $\texttt{Hom}(-, G)$

get exact sequence

$$0 \to \texttt{Hom}(C, G) \to \texttt{Hom}(B, G) \to \texttt{Hom}(A, G) \to \text{Ext}(C, G) \to \text{Ext}(B, C) \to \text{Ext}(A, G) \to 0$$

- if C is free or G injective, then $\text{Ext}(C, G) = 0$
- if this is a free resolution of C, then

$$0 \to \texttt{Hom}(C, G) \to \texttt{Hom}(B, G) \to \texttt{Hom}(A, G) \to \text{Ext}(C, G) \to 0$$

is exact

– this is the way to compute Ext

example:

$\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$

- $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ free resolution
- apply $\texttt{Hom}(-, \mathbb{Z})$
- $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to 0$

– $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$

example:

$\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$

- $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ free resolution
- apply $\texttt{Hom}(-, \mathbb{Q})$
- $0 \to \mathbb{Q} \xrightarrow{n} \mathbb{Q} \to \text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \to 0$

– $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \cong 0$

study relation between $H(C)$ and $H(\texttt{Hom}(C, G))$

Lemma 4.24. *If C consists of free groups, then for every n there is a split exact sequence*

$$0 \to \text{Ext}(H_{n-1}(C), G) \to H^n(\texttt{Hom}(C, G)) \to \texttt{Hom}(H_n(C), G) \to 0$$

This exact sequence is functorial in C.

Proof.

- $0 \to B_n \to Z_n \to H_n(C) \to 0$ is exact
- is a free resolution of $H_n(C)$ (subgroups of free abelian groups are free abelian)
- apply $\mathtt{Hom}(-, G)$

$$0 \to \mathtt{Hom}(H_n(C), G) \to \mathtt{Hom}(Z_n, G) \xrightarrow{i_n} \mathtt{Hom}(B_n, G) \to \mathrm{Ext}(H_n(C), G) \to 0$$

have exact sequence of vertical chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial} & B_{n-2} & \longrightarrow & 0 \end{array}$$

since B_n are free - horizontal sequences split

- applying $\mathtt{Hom}(-, G)$ gives exact sequence of vertical chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathtt{Hom}(B_{n-2}, G) & \longrightarrow & \mathtt{Hom}(C_{n-1}, G) & \xrightarrow{\partial} & \mathtt{Hom}(Z_{n-1}, G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathtt{Hom}(B_{n-1}, G)A & \longrightarrow & \mathtt{Hom}(C_n, G) & \xrightarrow{\partial} & \mathtt{Hom}(Z_n, G) & \longrightarrow & 0 \end{array}$$

Snake Lemma

$$\therefore^{i_n}. \mathtt{Hom}(B_{n-2}, G) \to H^{n-1}(\mathtt{Hom}(C, G)) \to \mathtt{Hom}(Z_{n-1}, G) \xrightarrow{i_{n-1}} \mathtt{Hom}(B_{n-1}, G) \to H^n(\mathtt{Hom}(C, G)) \to \mathtt{Hom}(Z_n,$$

conclude

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(\mathtt{Hom}(C, G)) \to \mathtt{Hom}(H_n(C), G) \to 0$$

get split map $s : C_n \to Z_n$

- get map $\bar{s} : C_n \to H_n(C)$
- get chain map $\bar{s} : C \to H(C)$

– check compatibility with differential

– $0 = \partial \bar{s}(c)$

– $\bar{s}(\partial c) = 0$ (since $\partial c \in Z_{n-1}$ and $s(\partial c) = \partial c$.

get map

$\mathtt{Hom}(H(C), G) \to \mathtt{Hom}(C, G)$

finally

$H(\mathtt{Hom}(H(C), G)) \to H(\mathtt{Hom}(C, G))$

this map splits the sequences

functoriality: exercise

□

define functor

$$C_{\mathtt{sing}}(-,-;G) : \mathbf{Top}^{2,\mathrm{op}} \to \mathbf{Ch}$$

-

$$C_{\mathtt{sing}}(-,-;G) := \mathtt{Hom}(C^{\mathtt{sing}}(-,-), G)$$

- $H_{\mathtt{sing}}(-,-,G) := H(C_{\mathtt{sing}}(-,-;G)) : \mathbf{Top}^{2,\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}-\mathrm{gr}}$ -

Definition 4.25. *The functor $H_{\mathtt{sing}}(-,-,G)$ is called the singular cohomology with coefficients in G*

(X, A) - a pair

$$0 \to C^{\mathtt{sing}}(X) \to C^{\mathtt{sing}}(X) \to C^{\mathtt{sing}}(X, A) \to 0$$

exact

$$0 \to C_{\mathtt{sing}}(X, A; G) \to C_{\mathtt{sing}}(X; G) \to C_{\mathtt{sing}}(A; G) \to 0$$

is exact (since $C^{\mathtt{sing}}$ consists of free groups).

get natural long exact sequence

$$H_{\mathtt{sing}}(X, A; G) \to H_{\mathtt{sing}}(X; G) \to H_{\mathtt{sing}}(A; G) \xrightarrow{\delta} H_{\mathtt{sing}}(X, A; G)[1]$$

(use cohomological grading)

Definition 4.26. *The pair $(H_{\mathtt{sing}}(-,-;G), \delta)$ is called the singular cohomology with coefficients in G.*

study properties:

let (X, A) be in $\mathbf{Top}^2$

Lemma 4.27 (universal coefficients)**.** *For every n in $\mathbb{N}$ we have a natural exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}^{\mathtt{sing}}(X,A), G) \to H^n_{\mathtt{sing}}(X,A;G) \to \mathtt{Hom}(H_n(X,A),G) \to 0 \ .$$

The sequence splits.

Lemma 4.28 (Homotopy invariance)**.** *$H_{\mathtt{sing}}(-,-;G)$ is homotopy invariant*

Proof.

X a space

- must show: $[0,1] \times X \to X$ induces an isomorphism
- get map of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ext}(H_{n-1}^{\mathtt{sing}}(X),G) & \to & H^n_{\mathtt{sing}}(X,G) & \to & \mathtt{Hom}(H_n^{\mathtt{sing}}(X),G) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Ext}(H_{n-1}^{\mathtt{sing}}([0,1]\times X),G) & \to & H^n_{\mathtt{sing}}([0,1]\times X,G) & \to & \mathtt{Hom}(H_n^{\mathtt{sing}}([0,1]\times X),G) & \to & 0 \end{array}$$

outer vertical maps are iso's

- middle vertical map is iso, too

□

Lemma 4.29 (Excision)**.** *For every pair (X, A) and subset U of X with $\bar{U} \subset \operatorname{int}(A)$ the map $H_{\mathtt{sing}}(X,A;G) \to H_{\mathtt{sing}}(X \setminus U, A \setminus U; G)$ is an isomorphism.*

Proof. analogous to homotopy invariance □

Lemma 4.30 (additivity)**.** *For a family $(X_i, A_i)_{i \in I}$ in $\mathbf{Top}^2$ we have an isomorphism (induced by the family of inclusions)*

$$H_{\mathtt{sing}}(X,A;G) \to \prod_{i \in I} H_{\mathtt{sing}}(X_i, A_i; G)$$

an isomorphism.

Proof.

use:

$\mathrm{Ext}(\bigoplus_{i\in I} A_i, G) \cong \prod_{i\in I} \mathrm{Ext}(A_i, G)$

- resolve A_i freely
- add up resolution
- apply $\mathtt{Hom}(-, G)$
- it turns sums into products
- kernel of a product of maps is product of kernels

then use universal coefficient formula $\square$

Lemma 4.31 (Exactness)**.** *For every pair (X, A) in $\mathbf{Top}^2$ we have a natural long exact sequence*

$$H_{\mathtt{sing}}(X, A; G) \to H_{\mathtt{sing}}(X; G) \to H_{\mathtt{sing}}(A; G) \xrightarrow{\delta} H_{\mathtt{sing}}(X, A; G)[1]$$

calculations

- $H_{\mathtt{sing}}(S^n; G) \cong G[0] \oplus G[-n]$
- $H_{\mathtt{sing}}(\mathbb{CP}^n; G) \cong \prod_{k=0}^{n} G[-2k]$

- $H_{\mathtt{sing}}(\mathbb{RP}^3; \mathbb{Z})$

use universal coefficients

- $H^0_{\mathtt{sing}}(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}$
- $H^1_{\mathtt{sing}}(\mathbb{RP}^3; \mathbb{Z}) \cong 0$ (since $\mathtt{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$)
- $H^2_{\mathtt{sing}}(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (contribution of $\mathrm{Ext}(H_1 ...)$)

$H^3_{\mathtt{sing}}(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}$ (contribution of $\mathrm{Ext}(H_1 ...)$

zero else

Theorem 4.32 (de Rham theorem)**.** *For a smooth manifold M there is a natural isomorphism*

$$H_{dR}(M) \cong H_{\mathtt{sing}}(M; \mathbb{R})$$

- kann in dieser Vorlesung nicht bewiesen werden

– Garbentheorie

– CW-Zerlegung

$\sigma : \Delta^n \to M$ may be smooth

- $\partial_i \sigma$ is still smooth etc

$\mathtt{sing}(M)^\infty$ - subcomplex of smooth simplices

Lemma 4.33. *The map $C(\mathtt{sing}(M)^\infty) \to C(\mathtt{sing}(M))$ is a quasi-isomorphism*

conclude

$\mathtt{Hom}(C(\mathtt{sing}(M), \mathbb{R}) \to \mathtt{Hom}(C(\mathtt{sing}(M)^\infty), \mathbb{R})$ is a quasi-isomorphism

integration map $I : \Omega(M) \to \mathtt{Hom}(C(\mathtt{sing}(M)^\infty), \mathbb{R})$

$I(\omega)(c) = \sum_{\sigma \in \mathtt{sing}(M)^\infty} c(\sigma) \int_{\Delta^n} \sigma^* \omega$

chain map

$$\begin{aligned}
I(d\omega)(c) &= \sum_{\sigma \in \mathtt{sing}(M)^\infty} c(\sigma) \int_{\Delta^n} \sigma^* d\omega \\
&= \sum_{\sigma \in \mathtt{sing}(M)^\infty} c(\sigma) \int_{\Delta^n} d\sigma^* \omega \\
&= \sum_{\sigma \in \mathtt{sing}(M)^\infty} c(\sigma) \int_{\Delta^{n-1}} \sum_{i=0}^{n} (-1)^n \partial_i \sigma^* \omega \\
&= I(\omega)(\partial c) \\
&= (\partial I(\omega))(c)
\end{aligned}$$

get signs right

Proposition 4.34. *The integration map is a quasi-isomorphism.*

4.8 More on homology of manifolds

fix ring R

abbreviate $H = H^{\mathtt{sing}}(-; R)$

- H takes values in $\mathbb{Z}$-graded R-modules

- $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong R$

- homeo $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ acts by multiplication by $\deg(f) \in \{1, -1\}$

M - manifold

- $n := \dim(M)$

- for m in M: $H(M, M \setminus \{m\}) \cong R$ as R-module (non canonically)

define covering $\tilde{M} \to M$ with fibre R as follows:

- set: $\tilde{M} := \bigsqcup_{x \in M} H_n(M, M \setminus \{m\})$

- projection: $\tilde{M} \to$ sends c in $H_n(M, M \setminus \{m\})$ to m

– is a bundle of free rank-one R-modules

- topology by local trivializations:

- closed disc D in M

- trivialize $H_n(D, \partial D) \times \mathrm{int}(D) \xrightarrow{\cong} p^{-1}(\mathrm{int}(D))$

– $(c, m) \mapsto r_m(c)$

— restriction map $r_m : H_n(D, \partial D) \xrightarrow{\cong} H_n(M, M \setminus \mathrm{int}(D)) \xrightarrow{\cong} H_n(M, M \setminus \{m\})$

- check: transition maps are locally constant by homotopy invariance

A closed in M

- $c \in H_n(M, M \setminus A)$

- get continuous section $A \to \tilde{M}$ setting $c(a) := r_a(c)$

Definition 4.35. *A homological R-orientation of M is a class $[M]_R$ in $H_n(M)$ such that $r_m([M]_R)$ is a generator of the R-module $H_n(M, M \setminus \{m\})$ for every m in M.*

Definition 4.36. *M is homologically R-orientable if there exists a homological R-orientation of M.*

Lemma 4.37. *The following are equivalent:*

1. *$\tilde{M} \to M$ is trivial.*
2. *$\tilde{M} \to M$ has a global section which is a generator in every fibre.*

Proof. Exercise. □

Lemma 4.38. *If $R = \mathbb{Z}/2\mathbb{Z}$, then $\tilde{M} \to M$ is trivial.*

Proof. cover M by discs in charts

- transition maps are multiplication with ± 1
- use $1 = -1$ in $\mathbb{Z}/2\mathbb{Z}$ □

Lemma 4.39. *If $\mathbb{R} = \mathbb{Z}$, then $\tilde{M}$ is trivial iff M is orientable.*

Proof.

$\tilde{M}$ trivial

- choose global section $s \in \Gamma(M, \tilde{M})$
- choose atlas such that $s(x) = 1$ in every chart
- this atlas is oriented

M oriented

- choose oriented atlas by discs
- transition maps are all multiplication by 1
- hence $\tilde{M} \to M$ is trivial □

Want to show:

Theorem 4.40. *M is homologically R-orientable if and only if $\tilde{M}$ is trivial and M is compact.*

must interpolate between $H(M, \emptyset)$ and $H(M, M \setminus \{m\})$

consider $H(M, M \setminus A)$ for all closed subsets A

abbreviate $\Gamma(A) := \Gamma(A, \tilde{M})$

A in M closed

have map $J^A : H_n(M, M \setminus A) \to \Gamma(A)$

- $c \mapsto r_a(c)$

Lemma 4.41. *J^A takes values in sections with compact support $\Gamma_c(A, \tilde{M})$*

Proof.

$[[c]] \in H_n(M, M \setminus A)$

$|c|$ is compact

- claim: a in $A \setminus |c|$ implies $J^A([[c]])(a) = 0$

– this class is in the image.

– $H_n(|c|, |c| \setminus (|c| \cap \{a\})) \to H_n(M, M \setminus \{a\})$ and domain vanishes (equal to $H_n(|c|, |c|)$)

□

M topological manifold, $\dim(M) = n$

Proposition 4.42. *For every closed subset A of M we have:*

D(A,1) $H_i(M, M \setminus A) = 0$ *for* $i > n$

D(A,2) $J^A : H_n(M, M \setminus A) \to \Gamma_c(A, \tilde{M})$ *is an isomorphism.*

1. $D(A, j)$, $D(B, j)$, $D(A \cap B, j)$ imply $D(A \cup B, j)$

   Mayer-Vietoris for $(M \setminus (A \cap B), M \setminus A, M \setminus B)$

$$
\begin{array}{ccc}
H_{n+1}(M, M \setminus A) \oplus H_{n-1}(M, M \setminus B) & \xrightarrow{\cong} & 0 \\
\downarrow & & \downarrow \\
H_{n+1}(M, M \setminus (A \cap B)) & \xrightarrow{\cong} & 0 \\
\downarrow & & \downarrow \\
H_n(M, M \setminus (A \cup B)) & \xrightarrow{J^{A \cup B}} & \Gamma_c(A \cup B) \\
\downarrow & & \downarrow \\
H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) & \xrightarrow[J^A \oplus J^B]{\cong} & \Gamma_c(A) \oplus \Gamma_c(B) \\
\downarrow & & \downarrow \\
H_n(M, M \setminus (A \cap B)) & \xrightarrow[J^{A \cap B}]{\cong} & \Gamma_c(A \cap B)
\end{array}
$$

   conclude $D(A \cup B, 2)$ by Five Lemma

   for $D(A \cup B, 1)$ use segment of MV-sequence for $k \geq 1$

   $H_{n+k+1}(M, M \setminus (A \cap B)) \to H_{n+k}(M, M \setminus (A \cup B) \to H_{n+k}(M, M \setminus A) \oplus H_{n+k}(M, M \setminus B)$

2. $D(A, j)$ is true for A which are convex compact in some chart

   $H_i(M, M \setminus A) \xrightarrow{r_a} H_i(M, M \setminus \{a\})$ is iso for every a in A and i in $\mathbb{Z}$

   - use excision to reduce to A in $\mathbb{R}^n$

   - use homotopy equivalence $(\mathbb{R}^n, \mathbb{R}^n \setminus A) \simeq (\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})$

   - get $D(A, 1)$ and $D(A, 2)$

   – note $\Gamma_c(A) = \Gamma(A)$ since A compact

3. $D(A, j)$ is true for A in a chart domain with $A = K_1 \cup \cdots \cup K_r$ with K_i compact and convex (in this domain)

   - proof by induction on r

   - $r = 1$ done by 2.

   - step $r - 1 \Rightarrow r$

   - $B := K_1 \cup \cdots \cup K_{r-1}$

   - $C := K_r$

   - $B \cap C = (K_1 \cap K_r) \cup \cdots \cup (K_{r-1} \cap K_r)$ intersections still convex

   - have $D(B, j)$, $D(C, j)$ and $D(B \cap C, j)$ by induction hypothesis

   - apply step 1.

4. $D(A, j)$ are true for compact A in a chart domain

   - cover A by balls

   - finitely many suffice by compactness

   - A admits neighbourhood A' as in 3.

   - can make balls smaller

   – $A = \mathtt{colim}_{A'} A'$ (A' as above)

   – $\mathtt{colim}_{A'} H_i(M, M \setminus A') \cong H_i(M, M \setminus A)$ (here we use singular homology)

   – conclude $D(A, 1)$ from $D(A', 1)$

$$\begin{array}{ccc}
\mathtt{colim}_{A'} H_n(M, M \setminus A') & \xrightarrow{\cong} & H_n(M, M \setminus A) \\
{\scriptstyle \cong} \downarrow {\scriptstyle \mathtt{colim}_{A'} J^{A'}} & & \downarrow {\scriptstyle J^A} \\
\mathtt{colim}_{A'} \Gamma_c(A') & \xrightarrow{\ !\ } & \Gamma_c(A)
\end{array}$$

   ! is isomorphism

   - can remove c by compactness

- every element in $\Gamma(A)$ extends to a neighbourhood of A

- every two extensions coincide on a smaller neighbourhood

– for every a in A can extend s to s_a in $\Gamma(U(a))$ for open neighbourhood $U(a)$ of a

– choose finite set B in A such that $A \subseteq \bigcup_{a \in B} U(a)$ (compactness of A)

– $W := \{y \in \bigcup_{a \in B} U(a) | (s_a)_{|U(a) \cap U(a')} = (s_{a'})_{|U(a) \cap U(a')}$for all pairs a, a' in $B\}$

– $A \subseteq W$

– W is open (since sections are locally constant)

– have extension of s to $\tilde{s}$ in $\Gamma(W)$

– $\tilde{s}'$ second extension on W'

– $\{y \in W \cap W' \mid \tilde{s}(y) = \tilde{s}'(y)\}$ is open neighbourhood of A

conclude $D(A, 2)$

5. $D(A, j)$ is true for all compact subsets A

   have decomposition $A = A_1 \cup \dots \cup A_r$ such that A_r is compact in chart domain

   - apply induction by r as in 3.

6. $D(A, j)$ is true for $A = \bigcup_{i \in I} A_i$ with A_i compact such that there exists family of opens $(U_i)_{i \in I}$ with $A_i \subseteq U_i$ and $(\bar{U}_i)_{\in I}$ pairwise disjoint

   - $H(M, M \setminus A) \cong \bigoplus_{i \in I} H(M, M \setminus A_i)$ by additivity

   – cover M by $\bigcup_{i \in I} U_i$ and $M \setminus A$.

   – Mayer-Vietoris $H(M, M \setminus A) \cong H(\bigcup_{i \in I} U_i, \bigcup_{i \in I}(U_i \setminus A_i))$

   – now use additivity $H(\bigcup_{i \in I} U_i, \bigcup_{i \in I}(U_i \setminus A_i)) \cong \bigoplus_{i \in I} H(U_i, U_i \setminus A_i)$

   – now use excision in each summand $H(U_i, U_i \setminus A_i) \cong H(M, M \setminus A_i)$

   - $\Gamma_c(A) \cong \bigoplus_{i \in I} \Gamma_c(A)_i$ (here compact support is important to get the sum)

7. $D(A, j)$ is true for general A

   - choose compact exhaustion $K_1 \subseteq K_2 \subseteq \dots$ of M

   – can assume $K_i \subseteq \text{int}(K_{i+1})$

   - $A_i := A \cap (K_i \setminus \text{int}(K_{i-1})$

   - $A_0 = \emptyset$

   - $B := \bigcup_{i=2n} A_i$

   - $C := \bigcup_{i=2n} A_{i+1}$

- $D(B,j)$ $D(C,j)$ and $D(B\cap C,j)$ true by 6.
- use step 1. to conclude $D(A,j)$

$\dim(M)=n$

apply Theorem for $A=M$

Corollary 4.43. *$H_i(M)=0$ for $i>n$.*

Proof of theorem 4.40.

M is homologically R-orientable

- $[M]_R$ in $H_n(M;R)$ R-homological orientation
- $J^M([M]_R)\in\Gamma_c(M)$ generating in each fibre
- M is compact (since $J^M([M]_R)$ has compact support)
- $\tilde{M}$ is trivial (global section $J^M([M]_R)$ provides trivialization)

$\tilde{M}$ is trivial and M is compact

- use trivialization to choose global section $s\in\Gamma_c(M)$ which generates each fibre
- $[M]_R := J^{M,-1}(s)$ in $H_n(M,M\setminus M)=H_n(M)$ is R-homological orientation □

4.9 Eilenberg-Zilber

compatibility of H (take homology) with tensor product

Theorem 4.44 (Künneth formula for homology)**.** *Let C and C' be lower-bounded chain complexes. Then for every n in $\mathbb{N}$ we have a natural exact sequence*

$$0\to\bigoplus_{p+q=n}H_p(C)\otimes H_q(C')\to H_n(C\otimes C')\to\bigoplus_{p+q=n-1}\mathtt{Tor}(H_p(C),H_q(C'))\to 0\ .$$

This sequence is split.

X,Y - topological spaces

Theorem 4.45 (Eilenberg-Zilber)**.** *There is a natural chain homotopy equivalence*

$$C^{\mathtt{sing}}(X\times Y)\to C^{\mathtt{sing}}(X)\otimes C^{\mathtt{sing}}(Y)\ .$$

Corollary 4.46 (Künneth theorem). *For spaces X, X' and all n in $\mathbb{N}$ we have natural short exact sequences*

$$0 \to \bigoplus_{p+q=n} H_p^{\mathtt{sing}}(X) \otimes H_q^{\mathtt{sing}}(X') \to H_n(X \otimes X') \to \bigoplus_{p+q=n-1} \mathtt{Tor}(H_p^{\mathtt{sing}}(X), H_q^{\mathtt{sing}}(X')) \to 0\,.$$

These sequences split.

method for Eilenberg-Zilber: acyclic models

abstract version of the method used to show homotopy invariance and excision (barycentric subdivision)

Definition 4.47. *A category with models is a pair $(\mathcal{C}, \mathcal{M})$ of a category and a subset of objects $\mathcal{M}$ of $\mathcal{C}$.*

$(\mathcal{C}, \mathcal{M})$ - category with models

- $G : \mathcal{C} \to \mathbf{Ab}$ - a functor

Definition 4.48. *A basis for G is a family $(M_i)_{i \in I}$ in $\mathcal{M}$ and a family $(g_i)_{i \in I}$ with $g_i \in G(M_i)$ such that for every X in $\mathcal{C}$ the family $(G(f)(m_i))_{i \in I, f \in \mathtt{Hom}_{\mathcal{C}}(M_i, X)}$ is a basis of $G(X)$*

remark:

- have representable functor

$$R := \bigoplus_{i \in I} \mathbb{Z}[\mathtt{Hom}_{\mathcal{C}}(M_i, -)] : \mathcal{C} \to \mathbf{Ab}$$

- determined by $(M_i)_{i \in I}$

- datum of family $(m_i)_{i \in I}$ is equivalent to datum of natural transformation $\phi : R \to G$

– given (m_i)

— isomorphism sends $\sum_{i \in I} \sum_{f \in \mathtt{Hom}_{\mathcal{C}}(M_i, X)} n_{i,f}[i, f]$ to $\sum_{i,f} n_{i,f} G(f)(m_i)$ in $G(X)$

– given ϕ

— recover m_i by $m_i = \phi(i, \mathtt{id}_{M_i})$ in $G(M_i)$

Definition 4.49. *G is called free if G admits a basis.*

variant:

- $C : \mathcal{C} \to \mathbf{Ch}$
- C is called free, if C_n is free for all n in $\mathbb{N}$

example:

$\mathcal{C} = \{*\}$

$\mathcal{M} = \{*\}$

G free abelian group

- G is free functor $* \to \mathbf{Ab}$
- choose basis $(g_i)_{i \in I}$ of $G = G(*)$

- C in $\mathbf{Ch}$ is functor $C : * \to \mathbf{Ch}$
- if C_n is free for all n in $\mathbb{Z}$, then C is free

example

$\mathcal{C} = \mathbf{Top}$

$\mathcal{M} = \{\Delta^k \mid k \in \mathbb{N}\}$

$C_n^{\mathtt{sing}} : \mathbf{Top} \to \mathbf{Ab}$

- is free
- $\mathtt{id}_{\Delta^n}$ in $C_n(\Delta^n)$
- $(\mathtt{id}_{\Delta^n})$ is basis
- $C_n(X) \cong \bigoplus_{\sigma:\Delta^n \to X} \mathbb{Z} C_n(\sigma)(\mathtt{id}_{\Delta^n})$
- $C^{\mathtt{sing}} : \mathbf{Top} \to \mathbf{Ch}$ is free

consider functor $C : \mathcal{C} \to \mathbf{Ch}$ with $C_n = 0$ for $n < 0$

$(\mathcal{C}, \mathcal{M})$ - category with models

Definition 4.50. *C is acyclic in positive dimensions if $H_q(C(M)) = 0$ for every M in $\mathcal{M}$ and q in $\mathbb{N}$ with $q > 0$.*

example: $C^{\mathtt{sing}}$ is acyclic in positive dimension on $(\mathbf{Top}, \{\Delta^k | k \in \mathbb{N}\})$

$(\mathcal{C}, \mathcal{M})$ - category with models

Proposition 4.51. *Assume that $C, C' : \mathcal{C} \to \mathbf{Ch}$ are two functors such that:*

1. *C is free.*
2. *C' is acyclic in positive dimensions.*

Then we have the following assertions:

1. *Every natrual transformation $H_0(C) \to H_0(C')$ is induced by a natural transformation $C \to C'$.*
2. *Two natural transformations $t, t' : C \to C'$ such that $H_0(t) = H_0(t')$ are naturally chain homotopic.*

example:

C, C' - positive chain compelxes

- C - free

- C' - acyclic in positive degree

- $t : H_0(C) \to H_0(C')$ morphism

– by. Proposition 4.51 there exists a chain map $C \to C'$ which is unique up to homotopy. and induces the map t in degree-0 homology

Proof of Theorem 4.45 assuming Proposition 4.51.

- $\mathcal{C} :=$ **Top** $\times$ **Top**

- models $\{(\Delta^p, \Delta^q) \mid (p, q) \in \mathbb{N} \times \mathbb{N}\}$

- functors:

– $C : (X, Y) \mapsto C^{\mathtt{sing}}(X \times X)$

– $C' : (X, Y) \mapsto C^{\mathtt{sing}}(X) \otimes C^{\mathtt{sing}}(Y)$

claim: C and C' are free and acyclic in positive dimensions

- free:

– fix n in $\mathbb{N}$

– C_n:

— $d_n : \Delta^n \to \Delta^n \times \Delta^n$ - diagonal

— d_n in $C_n(\Delta^n, \Delta^n)$

— (d_n) is basis for C

—- $C_n(X, Y) \cong \bigoplus_{(\sigma_0, \sigma_1):\Delta^n \to X \times Y} \mathbb{Z}C(\sigma_0, \sigma_1)(d_n)$

– C'_n :

– $\mathtt{id}_{\Delta^p} \otimes \mathtt{id}_{\Delta^q} \in C_p(\Delta^p) \otimes C_q(\Delta^q) \subseteq C_n(C(\Delta^p) \otimes C(\Delta^q))$

– $(\mathtt{id}_{\Delta^p} \otimes \mathtt{id}_{\Delta^q})_{(p,q)\in\mathbb{N}\times\mathbb{N},p+q=n}$ is basis of C'_n

– $C'_n(X \times Y) = \bigoplus_{(p,q)\in\mathbb{N}\times\mathbb{N},p+q=n,(f_0,f_1):(\Delta^p,\Delta^q)\to(X,Y)} \mathbb{Z}C'(f_0, f_1)(\mathtt{id}_{\Delta^p} \otimes \mathtt{id}_{\Delta^q})$

- acyclic

– C

– $\Delta^p \times \Delta^q$ is contractible

— $H_k(C(\Delta^p, \Delta^q)) \cong H_k^{\mathtt{sing}}(\Delta^p \times \Delta^q) \cong 0$ for $k > 0$

– C'

– $H_k(C'(\Delta^p, \Delta^q)) \cong H_k(C(\Delta^p) \otimes C(\Delta^q)) = 0$ for $k > 0$

— use $H_k^{\mathtt{sing}}(\Delta^p) = 0$ for $k > 0$ and Künneth for chain complexes

$H_0(C) \cong H_0(C')$

- $H_0(C(X,Y)) \cong H_0^{\mathtt{sing}}(X \times Y) \cong \mathbb{Z}[\pi_0(X \times Y)] \cong \mathbb{Z}[\pi_0(X) \times \pi_0(Y)] \cong \mathbb{Z}[\pi_0(X)] \otimes \mathbb{Z}[\pi_0(Y)] \cong H_0(C'(X,Y))$

– use $\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$, $\mathbb{Z}[A \times B] \cong \mathbb{Z}[A] \otimes \mathbb{Z}[B]$, and Künneth

by Proposition 4.51 the iso $H_0(C) \to H_0(C')$ and its inverse is induced by natural transformations

- $s : C \to C'$ and $t : C' \to C$

– $H_0(s) \circ H_0(t) = H_0(\mathtt{id}_{C'})$ implies: $s \circ t$ is naturally homotopic to $\mathtt{id}_{C'}$

– $H_0(t) \circ H_0(s) = H_0(\mathtt{id}_C)$ implies: $t \circ s$ is naturally homotopic to $\mathtt{id}_C$

□

Proof of Propositioin 4.51. fix $\phi : H_0(C) \to H_0(C')$

- construct natural chain map $\Phi : C \to C'$

– by induction $\Phi_{X,n} : C_n(X) \to C'_n(X)$ for all X

$(m_i)_{i\in I_n}$ - basis of C_n

start $n = 0$

- for every i in I_0 the element m_i in $C_0(M_i)$ represents class $[m_i]$ in $H_0(C(M_i))$

– choose m_i' in $C_0'(M_i)$ in class $\phi[m_i]$ in $H_0(C'(M_i))$

– for X in $\mathcal{C}$

— for i in I_0 and $f : M_i \to X$ in define $\Phi_{X,0} : C_0(X) \to C_0'(X)$ such uniquely such that

$$\Phi_{X,0}(C_0(f)(m_i)) = C_0'(f)(m_i')$$

– Φ_0 is natural: $h : X \to X'$

– $\Phi_{X',0}(C_0(h)(C_0(f)(m_i))) = \Phi_{X',0}(C_0(hf)(m_i)) = C_0'(hf)(m_i') = C_0'(h)(C_0'(f)(m_i')) = C_0'(h)(\Phi_{X,0}(C_0(f)(m_i)))$

– have shown $\Phi_{X',0} \circ C_0(h) = C_0'(h) \circ \Phi_{X,0}$

– $\Phi_{X,0}$ preserves boundaries (since it realizes a map on homology)

step $n-1 \Rightarrow n$

- for all i in I_n do:

– $\partial\Phi_{M_i,n-1}(\partial m_i) = \Phi_{M_i,n-1}(\partial^2 m_i) = 0$

— C' is acyclic in positive degrees (or $\Phi_{M_i,0}(\partial m_i)$ is a boundary in case $n = 1$)

— choose m_i' in $C_n'(M_i)$ such that $\partial m_i := \Phi_{M_i,n-1}(\partial m_i)$

– for X in $\mathcal{C}$

— for i in I_n and $f : M_i \to X$ in define $\Phi_{X,n} : C_n(X) \to C_n'(X)$ uniquely such that

$$\Phi_{X,n}(C_n(f)(m_i)) = C_n'(f)(m_i')$$

– ensures as above: Φ_n is natural

– $\partial\Phi_{X,n}(C_n(f)(m_i)) = \partial C_n'(f)(m_i') = C_{n-1}'(f)(\partial m_i') = C_{n-1}'(f)(\Phi_{M_i,n-1}(\partial m_i)) = \Phi_{X,n-1}(C_{n-1}(f)(\partial m_i))$ $\Phi_{X,n-1}(\partial C_n(f)(m_i))$

– read off: $\partial\Phi_{X,n} = \Phi_{X,n-1}\partial$

- induction step finished:

consider transformations

Φ, Φ' given

$H_0(\Phi) = H_0(\Phi')$

construct $h : C \to C[1]$ $\partial h + h\partial = \Phi - \Phi'$

- $h_n : C_n \to C_{n+1}'$ by induction

start $n = 0$

for all i in I_0 choose κ_i in $C'_1(M_i)$ such that $\partial\kappa_i = \Phi_{M_i,0}(m_i) - \Phi'_{M_i,0}(m_i)$

for X in $\mathcal{C}$

- for i in I_0, $f : M_i \to X$ define $h_{X,0}$ uniquely such that

$$H_{X,0}(C_0(f)(m_i)) := C'_1(f)(\kappa_i)$$

naturality is clear

- $\partial h_{X,0}(C_0(f)(m_i)) = \partial C'_1(f)(\kappa_i) = C'_0(f)(\partial\kappa_i) = C'_0(f)(\Phi_{M_i,0}(m_i) - \Phi'_{M_i,0}(m_i)) = \Phi_{X,0}(C_0(f)(m_i)) - \Phi'_{X,0}(C_0(f)(m_i))$

- read off $\partial H_{X,0} = \Phi_{X,0} - \Phi'_{X,0}$

step $n - 1 \Rightarrow n$

- for all i in I_n do:

$\partial(\Phi_{M_i,n}(m_i) - \Phi'_{M_i,n}(m_i) - H_{M_i,n-1}(\partial m_i)) = \Phi_{M_i,n-1}(\partial m_i) - \Phi'_{M_i,n-1}(\partial m_i) - \partial H_{M_i,n-1}(\partial m_i) = 0$

- C' is acyclic in positive degrees

– choose κ_i in $C'_{n+1}(M_i)$ such that

$$\partial\kappa_i = \Phi_{M_i,n}(m_i) - \Phi'_{M_i,n}(m_i) - H_{M_i,n-1}(\partial m_i)$$

– for X in $\mathcal{C}$

— for i in I_n and $f : M_i \to X$ in define $H_{X,n} : C_n(X) \to C'_{n+1}(X)$ such uniquely such that

$$H_{X,n}(C_n(f)(m_i)) = C'_{n+1}(f)(\kappa_i)$$

– ensures as above: H_n is natural

– $\partial H_{X,n}(C_n(f)(m_i)) = \partial C'_{n+1}(f)(\kappa_i) = C'_n(f)(\partial\kappa_i) = C'_n(f)(\Phi_{M_i,n}(m_i) - \Phi'_{M_i,n}(m_i) - H_{M_i,n-1}(\partial m_i))) = \Phi_{X,n}(C_n(f)(\partial m_i)) - \Phi'_{X,n}(C_n(f)(m_i)) - H_{X,n-1}(\partial C_n(f)(m_i)))$

– read off: $\partial H_{X,n} + H_{X,n-1}\partial = \Phi_{X,n} - \Phi'_{X,n}$

- induction step finished:

$\square$

Proposition 4.52. *For three spaces X, Y, Z the the following diagram commutes up to natural chain homotopy:*

$$\begin{array}{ccc} C^{\mathtt{sing}}((X \times Y) \times Z) & \longrightarrow & C^{\mathtt{sing}}(X \times (Y \times Z)) \\ \downarrow & & \downarrow \\ (C^{\mathtt{sing}}(X) \otimes C^{\mathtt{sing}}(Y)) \otimes C^{\mathtt{sing}}(Z) & \longrightarrow & C^{\mathtt{sing}}(X) \otimes (C^{\mathtt{sing}}(Y) \otimes C^{\mathtt{sing}}(Z)) \end{array}$$

here the vertical maps are given by iterated Eilenberg-Zilber maps and the horizontal maps are the associators of the products

Proof.

$\mathcal{C} = \mathbf{Top}^3$

$\mathcal{M} = \{\Delta^p \times \Delta^q \times \Delta^r \mid p, q, r \in \mathbb{N}\}$

$C :=$ right-down composition

$C' :=$ down-right composition

□

Proposition 4.53. *For two spaces X, Y the following diagram commutes up to natural chain homotopy*

$$\begin{array}{ccc} C^{\mathtt{sing}}(X \times Y) & \longrightarrow & C^{\mathtt{sing}}(Y \times X) \\ \downarrow & & \downarrow \\ C^{\mathtt{sing}}(X) \otimes C^{\mathtt{sing}}(Y) & \longrightarrow & C^{\mathtt{sing}}(Y) \otimes C^{\mathtt{sing}}(X) \end{array} .$$

here the vertical maps are given by Eilenberg-Zilber maps and the horizontal maps are the symmetry constraints of the products

- note that in $\mathbf{Ab}^{\mathbb{Z}-\mathrm{gr}}$ or $\mathbf{Ch}$: $s(x \otimes y) := (-1)^{\deg(x)\deg(y)} y \otimes x$

Proof.

$\mathcal{C} = \mathbf{Top}^2$

$\mathcal{M} = \{\Delta^p \times \Delta^q \mid p, q, \in \mathbb{N}\}$

$C :=$ right-down composition

$C' :=$ down-right composition

□

its difficult to understand in general what this implies to homology

- simplifying assumption: take rational coefficients

– $\otimes\mathbb{Q}$ everyting

– $H = H(-;\mathbb{Q})$

– kills $\texttt{Tor}$-terms

– $H(X) \otimes H(Y) \overset{\cong}{\to} H(X \times Y)$

Corollary 4.54.

– get map $\Delta : H(X) \overset{\texttt{diag}}{\to} H(X \times X) \cong H(X) \otimes H(X)$

– coproduct

– coassociative

— $(\Delta \otimes \texttt{id}) \circ \Delta = (\texttt{id} \otimes \Delta) \circ \Delta$

– cocommutative

— $s \circ \Delta = \Delta$

– counit: $\omega : H(X) \to H(*) \cong \mathbb{Q}[0]$ induced by $X \to *$

— $(\texttt{id} \otimes \omega) \circ \Delta = \texttt{id}$

— $(\omega \circ \texttt{id}) \circ \Delta = \texttt{id}$

— obvious by functoriality

Corollary 4.55. *$(H(X), \Delta, \omega)$ is a cocommutative coalgebra.*

to understand this notion

apply $\texttt{Hom}(-, \mathbb{Q}[0])$

- get commutative algebra with 1

$f : H(X) \to H(X')$ a map

- Does it come from a map of spaces?

if so:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X \times X & \longrightarrow & X' \times X' \end{array}$$

- hence

-

$$\begin{array}{ccc} H(X) & \xrightarrow{\quad f \quad} & H(X') \\ \downarrow \Delta & & \downarrow \Delta \\ H(X)\otimes H(X) & \xrightarrow{f\otimes f} & H(X')\otimes H(X') \end{array}$$

- f is map of coalgebras

- x in $H_n(X)$
- Is there a map $S^n \to X$ such that $x = f_*[S^n]$
– such classes are called spherical
- note: $\Delta([S^n]) = [S^n] \otimes [*] + [*] \otimes [S^n]$
follows from counit contraints
- necessary condition: $\Delta(x) = x \otimes [*] + [*] \otimes x$
- Such elements are called primitive.

consider cohomology
fix commutative ring R
- $C_{\mathtt{sing}}(-, R) := \mathtt{Hom}(C^{\mathtt{sing}}(-), R)$
- $\times$-product
- $C_{\mathtt{sing}}(X, R) \otimes_R C_{\mathtt{sing}}(Y, R) \cong \mathtt{Hom}(C^{\mathtt{sing}}(X), R) \otimes \mathtt{Hom}(C^{\mathtt{sing}}(Y), R) \to \mathtt{Hom}(C^{\mathtt{sing}}(X) \otimes C^{\mathtt{sing}}(Y), R \otimes_R R) \cong \mathtt{Hom}(C^{\mathtt{sing}}(X) \otimes C^{\mathtt{sing}}(Y), R) \to \mathtt{Hom}(C^{\mathtt{sing}}(X \times Y), R)$

Definition 4.56. *The induced map in cohomology is the exterior product*

$$- \times - : H_{\mathtt{sing}}(X, R) \otimes H_{\mathtt{sing}}(Y, R) \to H_{\mathtt{sing}}(X \times Y, R)$$

properties:
- for x in $H_{\mathtt{sing}}(X)$, y in $H_{\mathtt{sing}}(Y)$ z in $H_{\mathtt{sing}}(Z)$
– $(x \times y) \times z = x \times (y \times z)$
– $s^*(x \times y) = (-1)^{\deg(x)\deg(y)} y \times x$

fix X
- $\cup : H_{\mathtt{sing}}(X) \otimes H_{\mathtt{sing}}(X) \xrightarrow{\times} H_{\mathtt{sing}}(X \times X) \xrightarrow{\mathtt{diag}_X^*} H_{\mathtt{sing}}(X)$
- $\epsilon : R[0] \cong H_{\mathtt{sing}}(*) \to H_{\mathtt{sing}}(X)$ induced by $X \to *$

Corollary 4.57. *$(H(X), \cup, \epsilon)$ is a commutative unital algebra in* $\mathbf{Mod}(R)^{\mathbb{Z}-\mathrm{gr}}$

example:

$H_{\mathtt{sing}}(S^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^2)$, $\deg(x) = n$

(nothing to show)

$H_{\mathtt{sing}}(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, $\deg(x) = 2$

$H_{\mathtt{sing}}(\vee_{i=0}^{n} S^{2i}; \mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[x_i]/(x_i^2)/\sim \overset{\text{as a graded group}}{\cong} \mathbb{Z}[x]/(x^{n+1})$, $\deg(x_i) = 2i$

$\sim$ identifies the degree-zero terms with one copy of $\mathbb{Z}$

Proof of the Künneth formula.

C, C' - free chain complexes

have exact sequence of vertical chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial=0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial=0} & & \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial} & B_{n-2} & \longrightarrow & 0 \end{array}$$

interpret this as short exact sequence of chain complexes

$$0 \to Z \to C \to B[-1] \to 0$$

tensoring with C' gives exact sequence of chain complexes

$$0 \to Z \otimes C' \to C \otimes C' \to B[-1] \otimes C' \to 0$$

Snake Lemma

- $\cdots \to H_n(Z \otimes C') \to H_n(C \otimes C') \to H_{n-1}(B \otimes C') \xrightarrow{\delta} H_{n-1}(Z \otimes C')$
- δ is natural inclusion

– cycle in $B \otimes C'$ is sum of $\partial c \otimes z'$ with z' cycle in C'

– apply definition of boundary

use Z_p and B_p are free

- $H_n(Z \otimes C') \cong \bigoplus_{p+q=n} Z_p \otimes H_q(C')$
- $H_n(B \otimes C') \cong \bigoplus_{p+q=n} B_p \otimes H_q(C')$

now use $0 \to B_p \to Z_p \to H_p(C) \to 0$

$$0 \to \mathtt{Tor}(H_p(C), H_q(C')) \to B_p \otimes H_q(C') \to Z_p \otimes H_q(C') \to H_p(C) \otimes H_q(C') \to 0$$

sup up over $p+q=n$

- $\mathtt{ker}(\delta) = \bigoplus_{p+q=n} \mathtt{Tor}(H_p(C), H_q(C'))$
- $\operatorname{coker}(\delta) = \bigoplus_{p+q=n} H_p(C) \otimes H_q(C')$
- gives short exact sequence as claimed

$\square$