Classifying spaces and spectra representing the K-theory of a graded C^*-algebra.

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1 Introduction

A fundamental tool in Index theory and in the theory of C^*-algebras is Kasparov's bivariant K-theory which associates to C^*-algebras A, B an abelian group $KK(A,B)$ which is contravariant in A and covariant in B; here A, B are C^*-algebras over the ground field $\mathbb{F}=\mathbb{C}$ or $\mathbb{F}=\mathbb{R}$. These groups generalize the K-theory groups $K_n(B)$ of a C^*-algebra B which can be expressed as $K_n(B)=KK(\mathbb{F},C_0(\mathbb{R}^n,B))$, where $C_0(\mathbb{R}^n,B)$ is the C^*-algebra of continuous functions $\mathbb{R}^n\to B$ vanishing at infinity. An element of $KK(A,B)$ might be thought of as a "generalized" map from A to B. This point of view is motivated by the existence of a pairing constructed by Kasparov

$$KK(A,B)\otimes KK(B,C)\to KK(A,C), \tag{1.1}$$

which allows us to "compose" these generalized maps. Also, using the Kasparov product, it is not hard to associate to an element of $\phi\in KK(A,B)$ a homomorphism $\phi_*\colon K_n(A)\to K_n(B)$, which should be thought of as the map induced by ϕ on K-theory.

In this paper we associate to a $\mathbb{Z}/2$-graded σ-unital C^*-algebra[1] A a spectrum $\mathbf{K}(A)$ whose homotopy groups $\pi_n(\mathbf{K}(A))$ are the K-theory groups $K_n(A)$. Such spectra have been constructed before and are used e.g. to give a description of the assembly map or the Baum-Connes map in homotopy theoretic terms (cf. [17], [5], [19]). The point of the present paper is to show that these spectra can be constructed in a very desirable category of spectra, namely the category of orthogonal spectra[2]. In particular, we will show that $\mathbf{K}(A)$ for a C^*-algebra A over $\mathbb{F}=\mathbb{C},\mathbb{R}$ is a module spectrum over the commutative ring spectrum $\mathbf{K}(\mathbb{F})$ in this category of spectra.

A crucial feature of the category of orthogonal spectra is that if R is a commutative orthogonal ring spectrum, then the category of left R-modules has an internal Hom-functor, i.e. to R-module spectra M, N we can associate an R-module spectrum $Hom_R(M,N)$ with the formal properties we expect in the analoguous case of the category of modules over a commutative ring R, where $Hom_R(M,N)$ is the R-module of R-linear homomorphisms $M\to N$.

[1]All C^*-algebras in the paper will be required to be σ-unital.
[2]cf. Section 4

Similarly one can also define the graded set of homotopy classes of R-module maps $[M, N]_R$. In particular, for $\mathbb{Z}/2$-graded (σ-unital) C^*-algebras A, B over $\mathbb{F} = \mathbb{C}, \mathbb{R}$, we can define

$$\mathbf{K}_*(A, B) = [\mathbf{K}(A), \mathbf{K}(B)]_{\mathbf{K}(\mathbb{F})}.$$

One can show that the graded group $\mathbf{K}_*(A, B)$ has some nice formal properties: A short exact sequence of C^*-algebras in the A- or B-slot induces long exact sequences; the composition of $\mathbf{K}(\mathbb{F})$-module maps induces a product analogous to the Kasparov product (1.1). Indeed, there is a natural homomorphism from $KK(A, B)$ to $\mathbf{K}_*(A, B)$ for any A and B, which for $A = \mathbb{F}$ is an isomorphism. In this paper however we will not prove these properties but concentrate on constructing the orthogonal spectra $\mathbf{K}(A)$. Along the way we obtain various models for spaces which represent $K_*(A)$.

The construction of $\mathbf{K}(A)$ as well as the proofs heavily depend on elements of KK-theory. However, all facts we need can be extracted from the sources [4] and [13], resp. [18]. We also could have worked with other (more recent) approaches to K-homology. For example we could have used E-theory which has been developed over the past years, - or we could have used a $\mathbb{Z}/2$-graded analogue of the quasihomomorphism picture for KK-theory. The latter will be carried out in [12].

The paper is organized as follows. First we focus on ordinary (bounded) KK-theory to obtain a "space of bounded Kasparov modules" $\mathbf{F}^+_{H(A)}$ which represents $K_*(A)$. Essentially we want to use these spaces to construct the spectrum $\mathbf{K}(A)$ out of them. The structure maps of that spectrum then would correspond to certain (exterior) Kasparov products. These products, however, can be described much more easily in the unbounded KK-theory picture introduced by Baaj and Julg [3]. For that reason we replace the spaces $\mathbf{F}^+_{H(A)}$ by corresponding "subspaces of unbounded Kasparov modules"

$$\mathbf{R}^+_{H(A)} \subset \mathbf{F}^+_{H(A)}$$

(which are homotopy equivalent to $\mathbf{F}^+_{H(A)}$) and use these subspaces to define the orthogonal spectrum $\mathbf{K}(A)$.

`Conventions:`

- Tensor products are taken with respect to $\mathbb{F}$.
- The tensor products of C^*-algebras always will stand for the spatial tensor product.

2 The bounded picture

Given a C^*-algebra A as above and a countably generated $\mathbb{Z}/2$-graded Hilbert A-module E let $B(E)$ denote the C^*-algebra of bounded operators on E (being

A-equivariant with respect to the right action of A and admitting an adjoint with respect to the A-valued scalar product). The ideal $K(E) \subset B(E)$ of the compact operators is the norm closure of the algebra generated by all finite rank operators (cf. Blackadar [4, 13.2.3]).

Definition 2.1 *Let $\mathbf{F}_E$ be the subset of $B(E)$ of all odd operators F satisfying $||F|| \leq 1, F^* = F$ and $F^2 - 1 \in K(E)$. We give $\mathbf{F}_E$ the weakest topology such that the maps*

$$\begin{array}{lcl} F \in \mathbf{F}_E & \longmapsto & F\psi \in E, \quad \forall \psi \in E \\ F \in \mathbf{F}_E & \longmapsto & F^2 \in B(E) \end{array}$$

are continuous with respect to the norm topology of E and $B(E)$, respectively. Thus, the topology on $\mathbf{F}_E$ is the smallest one containing the strict topology and such that map $c : F \mapsto F^2 - 1$ is norm continuous.

Given a σ-compact Hausdorff space X let XE denote the Hilbert XA-module of continuous functions $X \to E$ vanishing at infinity, equipped with the XA-action coming from E.[3]

Lemma 2.2 *An operator $F \in \mathbf{F}_{XE}$ is the same as a strictly continuous family of operators $\{F_x \in F_E\}_{x \in X}$ such that $x \mapsto F_x^2 - 1$ is norm continuous and vanishes at infinity.*

PROOF : Since F commutes with the action of XA we conclude that F is given by a family $\{F_x\}_{x\in X}$ in $B(E)$ satisfying $||F_x|| \leq 1$ and $F_x^* = F_x$ for all $x \in X$. Now let $\psi \in E$, and for a given $x_0 \in X$ let us choose a function χ such that $\chi(x) \equiv \psi$ for all x in a neighborhood of x_0. In the neighborhood we then have $F_x\psi = F_x\chi(x) = (F\chi)(x)$. It follows that the function $x \mapsto F_x\psi$ is continuous at x_0. Doing this for all $x_0 \in X$ we conclude that the function $x \mapsto F_x\psi$ is continuous on the whole space X, i.e. the family $\{F_x\}$ is strictly continuous. Moreover we have $K(XE) = XK(E)$, i.e. a compact operator $K \in K(XE)$ is given by a norm continuous family of compact operators $\{K_x \in K(E)\}$ vanishing at infinity. Therefore the function $x \mapsto F_x^2 - 1$ is norm continuous and vanishes at infinity. $\diamond$

Given spaces X and Y let $C(X,Y)$ denote the mapping space, equipped with the compact-open topology.

Proposition 2.3 *For σ-compact Hausdorff spaces X there is a pullback square of topological spaces*

$$\begin{array}{ccc} \mathbf{F}_{XE} & \xrightarrow{c} & K(XE) \\ {\scriptstyle r}\downarrow & & \downarrow \\ C(X, \mathbf{F}_E) & \xrightarrow{c_*} & C(X, K(E)) \end{array},$$

[3] Note that the C^*-algebra $C_0(X)$ of functions vanishing at infinity is σ-unital if and only if the space X is σ-compact.

where r is given by $F \mapsto \{F_x\}$ (cf. Lemma 2.2) and c is given by $c(F) = F^2 - 1$. Therefore $c_* r(F)$ is given by the family $\{F_x^2 - 1\} \in K(E)$.

Note, that the canonical map $K(XE) = XK(E) \to C(X, K(E))$ is a homeomorphism, if X is compact. Therefore r is a homeomorphism for compact X.

PROOF : It follows from the definition of $\mathbf{F}_E$ that $C(X, \mathbf{F}_E)$ is a subset of $C(X, B(E))$. It carries the weakest topology such that the maps

$$\begin{array}{rcl} C(X, \mathbf{F}_E) & \to & C(X, B_1(E)_\beta) \\ C(X, \mathbf{F}_E) & \stackrel{c_*}{\to} & C(X, K(E)) \end{array}$$

are continuous, where $B_1(E)_\beta$ denotes the closed 1-ball in $B(E)$ equipped with the strict topology. From [20, 15.2.12], [1, 3.4], [14, 8.1] and [20, 2.J] it follows that there is a homeomorphism $B_1(XE)_\beta \cong C(X, B_1(E)_\beta)$. It follows that the compact-open topology on $C(X, B_1(E)_\beta)$ restricted to $r(\mathbf{F}_{XE})$ coincides with the restricted strict topology. The proposition now follows from the definition of the topology of $\mathbf{F}_{XE}$. $\diamond$

In the following we will use the Fredholm picture of KK-theory, i.e we associate to $F \in \mathbf{F}_E$ the Kasparov module (E, ϕ, F) with $\phi : \mathbb{F} \to B(E)$ unital (i.e. $\phi(1) = 1_{B(E)}$). Since the map ϕ canonically depends on E we will omit it from the notation in the following.

Let $H = H^0 \oplus H^1$ be a $\mathbb{Z}/2$-graded separable Hilbert space satisfying $\dim H^0 = \dim H^1 = \infty$. For example we may put $H = L^2([0,1]) \otimes Cl(1)$ with $Cl(1)$ the C^*-algebra generated by one odd element e satisfying $e = e^*$ and $e^2 = 1$. Given a C^*-algebra A we define $H(A) = H \otimes A$. We fix an operator $E_A \in B(H(A))$ which under an identification as above corresponds to left multiplication by e. Note that E_A qualifies as a reasonable basepoint for $F_{H(A)}$, and that the pair $(H(A), E_A)$ represents $0 \in K_0(A) = KK(\mathbb{F}, A)$.

Theorem 2.4 *There is a natural group structure on $\pi_0(\mathbf{F}_{H(A)})$, and associating to $F \in \mathbf{F}_{H(A)}$ its KK-class $[F] \in K_0(A)$ induces a natural group isomorphism*

$$b : \pi_0(\mathbf{F}_{H(A)}) \stackrel{\cong}{\to} K_0(A).$$

To prove the theorem we need the following lemma.

Lemma 2.5 *Let $U : H(A) \oplus H(A) \to H(A)$ be a unitary isomorphism. Then there is a path in $\mathbf{F}_{H(A)}$ connecting F and $U(F \oplus E_A)U^*$.*

PROOF : We use the identification $H(A) = L^2([0,1]) \otimes Cl(1) \otimes A$ and consider elements of $H(A)$ as L^2-functions on I with values in $Cl(1) \otimes A$. We then define an isomorphism $W : H(A) \to H(A) \oplus H(A), \psi \mapsto (W_1\psi, W_2\psi)$ by

$$(W_1\psi)(s) = \frac{1}{\sqrt{2}}\psi(s/2), \qquad (W_2\psi)(s) = \frac{1}{\sqrt{2}}\psi(s/2 + 1/2).$$

For $t \in (0,1]$ let v_t be the partial isometry on $L^2([0,1])$ given by

$$v_t(f)(s) = \begin{cases} \frac{1}{\sqrt{t}} f(s/t) & for \quad s \leq t \\ 0 & for \quad s > t \end{cases} .$$

We let $V_t = v_t \otimes 1 \otimes 1$ be the corresponding partial isometry on $H(A)$. Then

$$W^*(F \oplus E_A)W = V_{1/2} F V_{1/2}^* + (1 - V_{1/2} V_{1/2}^*) E_A.$$

The family

$$\Psi_t(F) = V_t F V_t^* + (1 - V_t V_t^*) E_A$$

for $t \in [1/2, 1]$ provides a path from F to $W^*(F \oplus E_A)W$ in $B(H(A))$. In fact, V_t is strongly continuous on $[1/2, 1]$. Thus, $\Psi_t(F)$ is strongly continuous. Moreover,

$$\Psi_t(F)^2 - 1 = V_t F^2 V_t^* - V_t V_t^* = V_t K V_t^*$$

for some compact K. But $V_t K V_t^*$ is norm continuous and compact.

We have shown that F and $W^*(F \oplus E_A)W$ can be connected by a path in $\mathbf{F}_{H(A)}$. Inserting U^*U we obtain that F and $W^*U^*U(F \oplus E_A)U^*UW$ can be connected by a path. Now we employ the fact, that the unitary group of $H(A) \oplus H(A)$ is connected in the strict topology. Thus, we find a family of unitaries Q_t connecting UW with the identity. Then $Q_t^* U(F \oplus E_A)U^* Q_t$ is a path connecting $W^*(F \oplus E_A)W$ and $U(F \oplus E_A)U^*$. $\diamond$

PROOF of Theorem 2.4 : Let $w : I \to \mathbf{F}_{H(A)}$ be a path in $\mathbf{F}_{H(A)}$ connecting elements F_0 and F_1. We can think of w as an element in $C(I, \mathbf{F}_{H(A)})$. The operator $r^{-1}(w) \in \mathbf{F}_{IH(A)}$ with r as in Proposition 2.3 provides a Kasparov homotopy between the Kasparov modules $(H(A), F_0)$ and $(H(A), F_1)$ by Proposition 2.3. Hence b is well-defined.
We now show surjectivity. Let us be given $x \in KK(\mathbb{F}, A)$ represented by a Kasparov module (E, F). If we add the degenerate Kasparov module $(H(A), E_A)$ to it we do not change the KK-class, i.e $x = [(E \oplus H(A), F \oplus E_A)]$. By the stabilization theorem ([4, 13.6.2], resp. [18, 2.1.4]) there is an unitary isomorphism $E \oplus H(A) \cong H(A)$. Hence $(E \oplus H(A), F \oplus E_A)$ is unitary equivalent to a Kasparov module $(H(A), F')$ and $b(F') = x$.
Next we consider injectivity. Let us be given $F_0, F_1 \in \mathbf{F}_{H(A)}$ with $b(F_0) = b(F_1) \in KK(\mathbb{F}, A)$, i.e. there is a Kasparov module (E, F) over IA which provides a Kasparov homotopy. The Kasparov module $(E \oplus H(IA), F \oplus E_{IA})$ then provides a Kasparov homotopy between $(H(A) \oplus H(A), F_0 \oplus E_A)$ and $(H(A) \oplus H(A), F_1 \oplus E_A)$. By the stabilization theorem there is a unitary right IA-linear isomorphism $U : E \oplus H(IA) \cong H(IA)$. Since U commutes with the right action of IA the operator U is given by a strictly continuous family of unitary operators $\{U_t\}_{t \in I}$, i.e. $U(F \oplus E_{IA})U^*$ is a Kasparov homotopy between $U_0(F_0 \oplus E_A)U_0^*$ and $U_1(F_1 \oplus E_A)U_1^*$. Injectivity then follows from the proposition and the lemma above.
We now define a semi-group structure on the set of path components of $\mathbf{F}_{H(A)}$ such that the map $b : \pi_0(\mathbf{F}_{H(A)}) \to K_0(A)$ becomes an isomorphism of groups.

The identity element of $\pi_0(\mathbf{F}_{H(A)})$ is the component containing E_A. Let us fix some unitary identification $U : H(A) \oplus H(A) \to H(A)$. Any will do and give the same additive structure, since any two of them are strictly homotopic. Let $[F_0]$, $[F_1]$ be the components represented by F_0, F_1. We define

$$[F_0] + [F_1] = [U(F_0 \oplus F_1)U^*].$$

Obviously, this definition is independent of the choice of the representatives of the components. Moreover, by the definition of the group structure on $KK(\mathbb{F}, A)$ we have $b([F_0] + [F_1]) = b([F_0]) + b([F_1])$, $b([E_A]) = 0$. This finishes the proof of Theorem 2.4. $\diamond$

We would like to point out that associating to a C^*-algebra A the space $\mathbf{F}_{H\otimes A}$ is not functorial in A. If $\varphi : A \to B$ is just a $*$-homomorphism the canonical map $S : (H \otimes A) \otimes_\varphi B \to H \otimes B$ is far from being an isomorphism in general. However, S is an isomorphism if $\varphi(A)B$ is dense in B. $*$-homomorphisms with this property we call essential. And indeed, associating $\mathbf{F}_{H\otimes A}$ to A is functorial with respect to essential $*$-homomorphisms. More precisely, if φ is an essential $*$-homomorphism the corresponding map $\varphi_* : \mathbf{F}_{H\otimes A} \to \mathbf{F}_{H\otimes B}$ is the map which sends F to $S(F \otimes_\varphi Id_B)S^{-1}$.

Since we are going to define a spectrum (in the sense of homotopy theory) we need to choose basepoints. Although E_A is a natural basepoint of $\mathbf{F}_{H(A)}$ it is not good enough for our purposes: we want the basepoint to be fixed under the conjugation action by unitary elements in $\mathbf{F}_{H(A)}$. To achieve a basepoint with this property we proceed as follows. Let us consider the subspace $\mathbf{U}_E$ of unitary elements in $\mathbf{F}_E$ and the corresponding cofibration sequence

$$\mathbf{U}_E \hookrightarrow \mathbf{F}_E \xrightarrow{p} \mathbf{F}_E/\mathbf{U}_E.$$

Note that $\mathbf{F}_E/\mathbf{U}_E$ naturally has a basepoint, namely $+ = \{\mathbf{U}_E\}$. To get better control over the neighborhoods of the basepoint we weaken the topology a little. (The essential reason for doing so is to obtain the pullback diagram (2.9).)

Definition 2.6 *Let the space $\mathbf{F}_E^+$ be the set $\mathbf{F}_E/\mathbf{U}_E$ equipped with the topology generated by the open sets in $\mathbf{F}_E - \mathbf{U}_E$ and the sets $\mathcal{U}_E^\varepsilon \cup \{+\}$ with*

$$\mathcal{U}_E^\varepsilon = \{F \in \mathbf{F}_E | \quad 0 < ||F^2 - 1|| < \varepsilon\} \quad (\varepsilon > 0).$$

Moreover we define (the continuous map) $\overline{p} : \mathbf{F}_E \to \mathbf{F}_E^+$ by $\overline{p}(F) = p(F)$.

Note that the map $c : \mathbf{F}_E \to K(E), F \mapsto F^2 - 1$ factorizes through $\mathbf{F}_E^+$, i.e. there is a well defined map $\overline{c} : \mathbf{F}_E^+ \to K(E)$ such that $c = \overline{c} \circ \overline{p}$. Note also, if $f : \mathbf{F}_E \to \mathbf{F}_E$ is map that satisfies $\|f(F)^2 - 1\| \le \|F^2 - 1\|$ then f induces a corresponding continuous map $f^+ : \mathbf{F}_E^+ \to \mathbf{F}_E^+$.

It follows from the next proposition and the corollary below that $\mathbf{F}_{H(A)}^+$ is a good substitute for $\mathbf{F}_{H(A)}$. To prove the proposition we will need the following.

Lemma 2.7 *Let $J = [-1,1]$ and let $u \in C(J,J)$ be an odd monotone function satisfying $|u(t)| \geq |t|$. Doing functional calculus on operators $F \in \mathbf{F}_E$ with the function u induces a homotopy equivalence $h(u) : \mathbf{F}_E \to \mathbf{F}_E$ which restricts to the identity on the subspace $\mathbf{U}_E$. Moreover $\|(h(u)(F))^2 - 1\| \leq \|F^2 - 1\|$, and the induced map $h(u)^+ : \mathbf{F}_E^+ \to \mathbf{F}_E^+$ also is a homotopy equivalence.*

PROOF : Let $F \in \mathbf{F}_E$. By the assumptions on u we have that $u(F)$ is odd, and it satisfies $\|u(F)\| \leq 1$ and $u(F) = u(F)^*$. Moreover, the assumptions on u also imply that $1 - F^2 \geq 1 - u(F)^2 \geq 0$; hence, since $1 - F^2 \in K(E)$, we have $1 - u(F)^2 \in K(E)$. Thus $u(F) \in \mathbf{F}_E$ for all $F \in \mathbf{F}_E$.

Now let us prove that the map $h(u)$ is continuous, i.e. we need to check that $F \mapsto u(F)^2$ is norm continuous, and that $F \mapsto u(F)$ is strictly continuous. Since $u(t)^2$ is even there is a function $v \in C([0,1],[0,1])$ such that $v(t^2) = u(t)^2$. Since F^2 is norm continuous, so is $v(F^2) = u(F)^2$, which shows the first assertion. To see the second one let us be given $\psi \in E$ and an $\varepsilon > 0$. Without loss of generality assume that $\psi \neq 0$. Using the Weierstraß approximation theorem we can find $n \in \mathbb{N}$ and coefficients $c_i \in \mathbb{C}$ for $0 \leq i \leq 2n$ such that

$$u(t) = g(t) + \sum_{k=0}^{n} t^{2k}(c_{2k} + c_{2k+1}t)$$

with g a function satisfying $\|g\|_\infty < \varepsilon/(4|\psi|)$. Put $M = \sup\{\|c_i\|,\ 0 \leq i \leq 2n\}$. Then, for $F, F_1 \in \mathbf{F}_E$ and any $0 \leq k \leq n$ we have (using $\|F\|, \|F_1\| \leq 1$)

$$\begin{aligned} &|F^{2k}(c_{2k} - c_{2k+1}F)\psi - F_1^{2k}(c_{2k} - c_{2k+1}F_1)\psi| \\ &\leq \ \|F^{2k}\|M|F\psi - F_1\psi| + \|F^{2k} - F_1^{2k}\|M(|\psi| + |F_1\psi|) \\ &\leq \ M|F\psi - F_1\psi| + k\|F^2 - F_1^2\|M2|\psi| \end{aligned}$$

Now, if F and F_1 are close in $\mathbf{F}_E$ then $|F\psi - F_1\psi|$ and $\|F^2 - F_1^2\|$ are small. Hence we can find a neighborhood $\mathcal{V}$ of F in $\mathbf{F}_E$ such that for any $0 \leq k \leq n$ and any $F_1 \in \mathcal{V}$ the expression on the left in the above inequality is bounded by $\varepsilon/2(n+1)$. Thus, if $F_1 \in \mathcal{V}$, we have

$$\begin{aligned} &|u(F)\psi - u(F_1)\psi| \\ &= \ |g(F)\psi - g(F_1)\psi + \sum\nolimits_{k=0}^{n} F^{2k}(c_{2k} + c_{2k+1}F)\psi - F_1^{2k}(c_{2k} + c_{2k+1}F_1)\psi| \\ &\leq \ 2\|g\|_\infty|\psi| + \sum\nolimits_{k=0}^{n} |F^{2k}(c_{2k} - c_{2k+1}F)\psi - F_1^{2k}(c_{2k} - c_{2k+1}F_1)\psi| \ \leq \varepsilon. \end{aligned}$$

This proves the second assertion.

With u also all functions $u_s(t) = (1-s)t + su(t)$, $s \in I$ satisfy the assumptions of the lemma. Thus we have a one parameter family of continuous maps $h(u_s) : \mathbf{F}_E \to \mathbf{F}_E$ which provide a homotopy $h : \mathbf{F}_E \times I \to \mathbf{F}_E$, given by $h(F,s) = u_s(F)$. The restriction of h for $s = 0$ and $s = 1$ are Id and $h(u)$ respectively. In particular, $h(u)$ is a homotopy equivalence. The assumptions on u also guarantee that $\|u_s(F)^2 - 1\| \leq \|F^2 - 1\|$ for all $s \in [0,1]$, and hence the induced maps $h(u_s)^+$ provide a homotopy between $Id_{\mathbf{F}_E^+}$ and $h(u)^+$. Thus the map $h(u)^+$ also is a homotopy equivalence. $\diamond$

Proposition 2.8 $p : \mathbf{F}_{H(A)} \to \mathbf{F}^+_{H(A)}$ *is a homotopy equivalence.*

PROOF : This follows essentially from the fact that $\mathbf{U}_{H(A)}$ is contractible. More precisely, let us extend the definition of Ψ_t in the proof of Lemma 2.5 to all $t \in [0, \infty)$ using

$$v_0 = 0, \quad v_t = 1 \in B(L^2([0,1])) \text{ for } t > 1.$$

We now define the continuous map $l : \mathbf{F}_{H(A)} \to \mathbb{R}, l(F) \mapsto \min(1/2, ||F^2 - 1||)$ and consider the homotopy

$$\Phi_t(F) = \Psi_{t+l(F)}(F).$$

Φ provides a homotopy from the identity of $\mathbf{F}_{H(A)}$ to the map Φ_0 which factors over $\mathbf{F}_{H(A)}/\mathbf{U}_{H(A)}$, i.e. we have a factorization

$$\mathbf{F}_{H(A)} \stackrel{p}{\to} \mathbf{F}_{H(A)}/\mathbf{U}_{H(A)} \stackrel{s}{\to} \mathbf{F}_{H(A)}.$$

We might ask if s is continuous if we regard it as map out of $\mathbf{F}^+_{H(A)}$, instead as a map out of $\mathbf{F}_{H(A)}/\mathbf{U}_{H(A)}$. In general this is not the case. However we can use the lemma above to deform s a little so it will become continuous as a map out of $\mathbf{F}^+_E$, as we will show now.

Choose a function u as in the previous lemma which in addition is locally constant around $1 \in J$, say it maps $[1-\varepsilon, 1]$ to 1 for a suitably chosen $\varepsilon > 0$. We now define

$$\Phi^u_t(F) = \Phi_t(u(F)).$$

By construction Φ^u_0 maps all $F \in \mathbf{F}_E$ with $||F^2 - 1|| < \varepsilon$ to E_A, hence Φ^u_0 has a factorization

$$\Phi^u_0 : \mathbf{F}_{H(A)} \xrightarrow{\overline{p}} \mathbf{F}^+_E \xrightarrow{\overline{s_u}} \mathbf{F}_{H(A)}.$$

Using the previous lemma we know that this composition is a homotopy equivalence (since $\Phi^u_0 = \Phi_0 \circ h(u)$, and $h(u)$ and Φ_0 are both homotopy equivalences). In particular, $\overline{p}$ has a homotopy left inverse.

On the other hand we have

$$||\Phi^u_t(F)^2 - 1|| \leq ||\Phi_t(F)^2 - 1|| \leq ||F^2 - 1||$$

for all $F \in \mathbf{F}_E$. Hence we have a continuous family of maps $(\Phi^u_t)^+ : \mathbf{F}^+_E \to \mathbf{F}^+_E$ which provides a homotopy between $(\Phi^u_1)^+ = h(u)^+$ and $(\Phi^u_0)^+ = \overline{p} \circ \overline{s_u}$. Since $h(u)^+$ is a homotopy equivalence $\overline{p}$ has a homotopy right inverse. $\diamond$

To investigate the higher homotopy groups of $\mathbf{F}_{H(A)}$ we first introduce some notation. Given a σ-compact space X and a pointed space Y we consider the mapping space $C_0(X, Y)$ of continuous functions "pointed at infinity". A function $f : X \to Y$ belongs to $C_0(X, Y)$ if and only if there is a continuous pointed extension $f : X^c \to Y$ from the one-point-compactification X^c to Y. Conversely, any continuous pointed map $f : X^c \to Y$ gives an element in $C_0(X, Y)$

by restriction. We use this correspondence to identify $C_0(X,Y)$ with the space $C_0(X^c,Y)$ of continuous pointed maps $X^c \to Y$. $C_0(X,Y)$ always is a subset of $C(X,Y)$, but in general it is not a subspace: E.g. if Y is a metric space then $C_0(X,Y)$ carries the topology of uniform convergence, while its image in $C(X,Y)$ carries the restricted topology of uniform convergence on compact sets.

If X is σ-compact Hausdorff we have injections

$$r : \mathbf{F}_{XE} \to C(X, \mathbf{F}_E) \quad \text{and} \quad C_0(X, \mathbf{F}_E^+) \to C(X, \mathbf{F}_E^+).$$

The map $p_* : C(X,\mathbf{F}_E) \to C(X,\mathbf{F}_E^+)$ given by composing a map in $C(X,\mathbf{F}_E)$ with $\overline{p} : \mathbf{F}_E \to \mathbf{F}_E^+$ then defines a map $\pi : \mathbf{F}_{XE} \to C_0(X,\mathbf{F}_E^+)$; and we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{F}_{XE} \arrow[r, "\pi"] \arrow[d, "r"'] & C_0(X,\mathbf{F}_E^+) \arrow[r] \arrow[d] & K(XE) \arrow[d] \\ C(X,\mathbf{F}_E) \arrow[r, "\overline{p}_*"] & C(X,\mathbf{F}_E^+) \arrow[r, "\overline{c}_*"] & C(X,K(E)). \end{tikzcd} \tag{2.9}$$

All squares in the diagram are pullback squares.

Lemma 2.10 *Let $f : \mathbf{F}_E \to \mathbf{F}_E$ be a map which satisfies $\|f(F)^2-1\| \leq \|F^2-1\|$ for all $F \in \mathbf{F}_E$. Then for any σ-compact Hausdorff space X the induced maps $f_* : C(X,\mathbf{F}_E) \to C(X,\mathbf{F}_E)$ and $f_*^+ : C(X,\mathbf{F}_E^+) \to C(X,\mathbf{F}_E^+)$ have continuous liftings*

$$\begin{aligned} \widetilde{f}_* &: \mathbf{F}_{XE} \longrightarrow \mathbf{F}_{XE}, \\ \widetilde{f}_*^+ &: C_0(X,\mathbf{F}_E^+) \longrightarrow C_0(X,\mathbf{F}_E^+). \end{aligned}$$

PROOF : The assumption $\|f(F)^2-1\| \leq \|F^2-1\|$ guarantees that $c_* \circ f_*$ lifts to a continuous map $\mathbf{F}_{XE} \to KXE$; hence, by Proposition 2.3, we get a map $\widetilde{f}_* : \mathbf{F}_{XE} \to \mathbf{F}_{XE}$, which is a lift of f_*. Using the right square in the pullback diagram above we analogously obtain a lift $\widetilde{f}_*^+$ of f_*^+. For the latter we also simply could have used the fact that $C_0(X,Y)$ is a covariant functor in Y: we then have $\widetilde{f}_*^+ = C_0(X,f^+)$. $\diamond$

One of the consequences of the previous lemma is the following: If $u \in C([-1,1],[-1,1])$ is as in Lemma 2.7 then the map $h(u)$ defined there induces homotopy equivalences

$$\begin{aligned} \widetilde{h(u)}_* &: \mathbf{F}_{XE} \longrightarrow \mathbf{F}_{XE}, \\ \widetilde{h(u)}_*^+ &: C_0(X,\mathbf{F}_E^+) \longrightarrow C_0(X,\mathbf{F}_E^+). \end{aligned}$$

In fact these maps are homotopic to the identity; corresponding homotopies to the identity are given by the maps $\widetilde{h(u_s)}_*$ and $\widetilde{h(u_s)}_*^+$ with $u_s \in C([-1,1],[-1,1])$ given by $u_s(t) = (1-s)t + su(t)$.

Proposition 2.11 $\pi : \mathbf{F}_{XH(A)} \to C_0(X, \mathbf{F}^+_{H(A)})$ *is a homotopy equivalence.*

PROOF : A homotopy inverse is given by

$$\sigma : C_0(X, \mathbf{F}^+_{H(A)}) \longrightarrow \mathbf{F}_{XH(A)}, \quad g \mapsto \overline{s_u} \circ g,$$

where $\overline{s_u} : \mathbf{F}^+_{H(A)} \to \mathbf{F}_{H(A)}$ is defined as in the proof of Proposition 2.8 and $\overline{s_u} \circ g$ is regarded as an element in $\mathbf{F}_{XH(A)}$ using Lemma 2.2.

To see that σ is a homotopy inverse of π we consider the homotopy

$$\Theta_t = (\Phi^u_t)_* : C(X, \mathbf{F}_{H(A)}) \to C(X, \mathbf{F}_{H(A)}), \quad f \mapsto \Phi^u_t \circ f$$

with $\Phi^u_t : \mathbf{F}_{H(A)} \to \mathbf{F}_{H(A)}$ as in the proof of 2.8. The family of maps Θ_t lifts to a continuous family $\widetilde{\Theta_t} : \mathbf{F}_{XH(A)} \to \mathbf{F}_{XH(A)}$. It provides a homotopy between $\widetilde{\Theta_1} = \widetilde{h(u)}_*$ and $\widetilde{\Theta_0} = \sigma \circ \pi$. Therefore σ is a homotopy left inverse of π.
On the other hand, using the family $(\Phi^u_t)^+ : \mathbf{F}^+_{H(A)} \to \mathbf{F}^+_{H(A)}$ we obtain the homotopy $\Theta^+_t = (\Phi^u_t)^+_* : C(X, \mathbf{F}^+_{H(A)}) \to C(X, \mathbf{F}^+_{H(A)})$ that lifts to a homotopy

$$\widetilde{\Theta^+_t} : C_0(X, \mathbf{F}^+_{H(A)}) \to C_0(X, \mathbf{F}^+_{H(A)}).$$

Here we have $\widetilde{\Theta^+_1} = \widetilde{h(u)^+_*}$, and $\widetilde{\Theta^+_0} = \pi \circ \sigma$. Hence σ also is a homotopy right inverse of π. $\diamond$

Corollary 2.12 *We have natural group isomorphisms*

$$K_n(A) \cong \pi_0(\mathbf{F}^+_{H(\mathbb{R}^n A)}) \cong \pi_0(\mathbf{F}^+_{\mathbb{R}^n H(A)}) \cong \pi_0(C_0(\mathbb{R}^n, \mathbf{F}^+_{H(A)})) \cong \pi_n(\mathbf{F}^+_{H(A)}).$$

PROOF : That the first map is an isomorphism follows from the definition of $K_n(A)$, Theorem 2.4 and Proposition 2.8. The second isomorphism is induced by the isomorphism $H(\mathbb{R}^n A) \cong \mathbb{R}^n H(A)$. The third map is an isomorphism by Proposition 2.11, and the last map is an isomorphism essentially by definition. Note that (for $n > 0$) we have used two a priori different group structures on $\pi_0(C_0(\mathbb{R}^n, \mathbf{F}^+_{H(A)}))$: namely the one given by the above theorem and the one induced by the pinching map $S^n \to S^n \vee S^n$. However it is not hard to see that the two agree. $\diamond$

3 The unbounded picture

Since the exterior intersection product will go into the definition of the ring spectrum that will represent $K_*(A)$ it is more convenient to use the Baaj-Julg picture of KK-theory (cf. Proposition 3.5). In addition (for later reference) we need to quote some basic facts from KK-theory.

3.1 Regular Operators

Recall that an A-linear, closed densely defined operator D on a Hilbert A-module E is called regular, if domD^* is dense and $(1 + D^*D)$ has dense image in E. If D is a regular operator, then $(1 + D^*D)^{-1} \in B(E)$ ([4, 13.3]).

Definition 3.1 *The set* $\mathbf{R}'_E$ *is defined by*

$$\mathbf{R}'_E = \{D \textit{ regular on } E \mid D^* = D, D \textit{ odd}, \; (1 + D^2)^{-1} \in K(E)\}.$$

We define the map $J : \mathbf{R}'_E \to \mathbf{F}_E$ by $J(D) = D(1 + D^2)^{-1/2}$. Baaj and Julg have shown that indeed $J(D) \in \mathbf{F}_E$ ([3, Prop. 2.2]). In particular this means that an operator $D \in \mathbf{R}'_E$ determines a class $\beta(D) = b(J(D)) \in KK(\mathbb{F}, A)$.

We can obtain D back from $J(D)$ by the formula

$$D = J(D)(1 - J^2(D))^{-1/2}.$$

Note that $1 - J^2(D) = (1 + D^2)^{-1}$ and therefore $dom(D^2) = im((1 + D^2)^{-1}) = dom((1 - J^2(D))^{-1}$. Since $dom(D^2) \subset dom(1 + D^2)^{1/2}$ we have

$$J(D)(1 - J^2(D))^{-1/2} = D(1 + D^2)^{-1/2}(1 + D^2)^{1/2} = D \qquad \text{on} \quad dom(D^2).$$

Since $dom(D^2)$ is a core for D (cf. [14], Lemma 9.2) we finally conclude that $J(D)(1 - J^2(D))^{-1/2} = D$.

Using this correspondence we define $\mathbf{R}_E$ as the subspace $\mathbf{R}_E = J(\mathbf{R}'_E) \subset \mathbf{F}_E$, and abusing notation we also use the letter J for the inclusion $J : \mathbf{R}_E \subset \mathbf{F}_E$. Note that $J(\mathbf{R}'_E) \cap \mathbf{U}_E = \emptyset$. Hence we equally well may regard $\mathbf{R}_E$ as a subspace of $\mathbf{F}^+_E$ (via $p \circ J : \mathbf{R}_E \to \mathbf{F}_E \to \mathbf{F}^+_E$).

Definition 3.2 *Let* $\mathbf{R}^+_E$ *be the pointed set* $\mathbf{R}^+_E = (\mathbf{R}_E \cup \{+\}, +)$ *equipped with the topology induced by* $J^+ : \mathbf{R}^+_E \to \mathbf{F}^+_E$*, the pointed extension of* $p \circ J$.

Proposition 3.3 J^+ *is a homotopy equivalence. In particular, there is a natural bijection* $\pi_0(\mathbf{R}^+_{H(A)}) \to KK(\mathbb{F}, A)$ *(given by* $b \circ p_*^{-1} \circ J^+_*$*).*

Proof : Since E is countably generated there is an even strictly positive element $K \in K(E)$ with $\|K\| \le 1$. For $t \in [0, 1/4]$ define $\Lambda_t : \mathbf{F}_E \to \mathbf{F}_E$ by

$$\Lambda_t(F) = (1 - t\|F^2 - 1\|K)F(1 - t\|F^2 - 1\|K).$$

Since $\|\Lambda_t(F)^2 - 1\| \le C\|F^2 - 1\|$ for some constant C we obtain an induced family $\Lambda^+_t : \mathbf{F}^+_E \to \mathbf{F}^+_E$. The family Λ^+_t satisfies $\Lambda^+_0 = Id$, $\Lambda^+_{1/4}(\mathbf{F}_E) \subset J^+(\mathbf{R}_E) \cup \mathbf{U}_E$ and $\Lambda^+_t(J^+(\mathbf{R}^+_E)) \subset J^+(\mathbf{R}^+_E), \forall t \in [0, 1/4]$. Hence it follows that $J^+ : \mathbf{R}^+_E \to \mathbf{F}^+_E$ is a homotopy equivalence. $\diamond$

Corollary 3.4 *For a σ-compact Hausdorff space X the map*

$$\mathbf{R}^+_{XH(A)} \to C_0(X, \mathbf{R}^+_{H(A)})$$

is a homotopy equivalence.

Proof : The map fits into the commutative diagram

$$\begin{array}{ccc} \mathbf{R}^+_{XH(A)} & \longrightarrow & C_0(X, \mathbf{R}^+_{H(A)}) \\ {\scriptstyle J^+}\Big\downarrow & & \Big\downarrow{\scriptstyle J^+_*} \\ \mathbf{F}^+_{XH(A)} & \longrightarrow & C_0(X, \mathbf{F}^+_{H(A)}). \end{array}$$

The vertical maps are homotopy equivalences by the proposition above. The lower horizontal map is a homotopy equivalence by Propositions 2.8 and 2.11.

Proposition 3.5 (Baaj-Julg [3], Théorème 3.2)
If E and E' are Hilbert modules over A and A' respectively, then the map

$$\mu : \mathbf{R}_E \times \mathbf{R}_{E'} \to \mathbf{R}_{E\otimes E'}, (D, D') \mapsto D \otimes 1 + 1 \otimes D'$$

induces the external Kasparov product $KK(\mathbb{F}, A) \times KK(\mathbb{F}, A') \to KK(\mathbb{F}, A\otimes A')$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathbf{R}_E \times \mathbf{R}_{E'} & \xrightarrow{\mu} & \mathbf{R}_{E\otimes E'} \\ {\scriptstyle \beta\times\beta}\Big\downarrow & & {\scriptstyle \beta}\Big\downarrow \\ KK(\mathbb{F}, A) \times KK(\mathbb{F}, A') & \xrightarrow{\otimes} & KK(\mathbb{F}, A \otimes A') \end{array}$$

Proposition 3.6 *The map $\mu : \mathbf{R}_E \times \mathbf{R}_{E'} \to \mathbf{R}_{E\otimes E'}$ is continuous and has a continuous pointed extension $\mu^+ : \mathbf{R}^+_E \wedge \mathbf{R}^+_{E'} \to \mathbf{R}^+_{E\otimes E'}$.*

Proof : First we show that μ is continuous. We write

$$\begin{aligned} 1 - J(\mu(D, D'))^2 &= (\mu(D, D')^2 + 1)^{-1} \\ &= (D^2 \otimes 1 + 1 \otimes (D')^2 + 1)^{-1} \\ &= (D^2 \otimes 1 + 1)^{-1} \left\{1 + (1 \otimes (D')^2)(D^2 \otimes 1 + 1)^{-1}\right\}^{-1}. \end{aligned}$$

The functions $f_\lambda(x) = x(1 + \lambda x)^{-1}$ are uniformly Lipschitz on $[0, 1]$ as λ varies in $[0, \infty)$. Thus, if $D, D_1 \in \mathbf{R}_E$ with $\|(D^2 + 1)^{-1} - (D_1^2 + 1)^{-1}\|$ being small, then so is $\|(D^2 \otimes 1 + 1 \otimes (D')^2 + 1)^{-1} - (D_1^2 \otimes 1 + 1 \otimes (D')^2 + 1)^{-1}\|$ uniformly with respect to D'. A similar statement holds if we vary D'.

Let now $\psi \in E \otimes E'$. We consider the function $\mathbf{R}_E \times \mathbf{R}_{E'} \to E \otimes E'$,

$$(D, D') \longmapsto (D \otimes 1 + 1 \otimes D') \left\{D^2 \otimes 1 + 1 \otimes (D')^2 + 1\right\}^{-1/2} \psi.$$

We have

$$(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}\psi =$$
$$\left\{1+1\otimes (D')^2(D^2\otimes 1+1)^{-1}\right\}^{-1/2}\left\{(D\otimes 1)(D^2\otimes 1+1)^{-1/2}\psi\right\}.$$

The functions $h_\lambda(x)=(1+\lambda x)^{-1/2}$ are uniformly Lipschitz on $[0,1]$ as λ varies in $[0,\infty)$. Thus, if D and D_1 are close in $\mathbf{R}_E$, then

$$|(D\otimes 1)(D^2\otimes 1+1)^{-1/2}\psi-(D_1\otimes 1)(D_1^2\otimes 1+1)^{-1/2}\psi|\quad \text{and}$$
$$||(1+1\otimes (D')^2(D^2\otimes 1+1)^{-1})^{-1/2}-(1+1\otimes (D')^2(D_1^2\otimes 1+1)^{-1})^{-1/2}||$$

are small uniformly with respect to D'. Hence so is

$$|(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}\psi-(D_1\otimes 1)\left\{D_1^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}\psi|.$$

We can also write

$$(D\otimes 1)\left\{(D^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}=$$
$$(D\otimes 1)\left\{(D^2\otimes 1)(1\otimes (D')^2+1)^{-1}+1\right\}^{-1/2}.$$

The functions $q_\lambda(x)=\lambda(\lambda^2x+1)^{-1/2}$ are uniformly Lipschitz on $[0,1]$ as λ varies in $(-\infty,\infty)$. Thus, if D' and D_1' are close in $\mathbf{R}_{E'}$, then

$$||(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}-(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D_1')^2+1\right\}^{-1/2}||$$

is small uniformly with respect to D. In particular so is

$$|(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D')^2+1\right\}^{-1/2}\psi-(D\otimes 1)\left\{D^2\otimes 1+1\otimes (D_1')^2+1\right\}^{-1/2}\psi|.$$

Doing the same arguments with D and D' interchanged we can eventually conlcude that $\mu:\mathbf{R}_E\times\mathbf{R}_{E'}\to\mathbf{R}_{E\otimes E'}$ is continuous.

It remains to show continuity in $+\in\mathbf{R}_E^+\wedge\mathbf{R}_{E'}^+$. Recall that a neighborhood basis for $+\in\mathbf{R}_E$ is given by the sets $V_E^\varepsilon\cup\{+\}$ with

$$V_E^\varepsilon=\left\{D\in\mathbf{R}_E|\quad ||1-J(D)^2||<\varepsilon\right\}\quad (\varepsilon>0).$$

We now introduce $V_{E,E'}^\varepsilon=V_E^\varepsilon\times\mathbf{R}_{E'}\cup\mathbf{R}_E\times V_{E'}^\varepsilon$. Continuity in $+\in\mathbf{R}_E^+\wedge\mathbf{R}_{E'}^+$ then follows from $\mu(V_{E,E'}^\varepsilon)\subset V_{E\otimes E'}^\varepsilon$, which we prove next.

We know that $D^2\otimes 1$ and $1\otimes (D')^2$ are non-negative commuting self-adjoint operators on $E\otimes E'$. We apply functional calculus to them using $f(x,y)=(1+x+y)^{-1}$. Then

$$1-J(\mu(D,D'))^2=f(D^2\otimes 1,1\otimes (D')^2)\geq 0.$$

Since $f(x,y)$ is decreasing in x and y for $x,y\geq 0$ we obtain

$$\min\left\{||f(D^2\otimes 1,0)||,||f(0,1\otimes (D')^2)||\right\}\geq ||f(D^2\otimes 1,1\otimes (D')^2)||.$$

However, $||f(D^2\otimes 1,0)||=||1-J(D)^2||$ and $||f(0,1\otimes (D')^2)||=||1-J(D')^2||$, and therefore we have $\mu(V_{E,E'}^\varepsilon)\subset V_{E\otimes E'}^\varepsilon$. $\diamond$

3.2 The Dirac- and the dual Dirac-element

The goal of this section is to provide certain weak homotopy equivalences (cf. Corollary 3.12) which go into the proofs of the next section. To establish them we will quote some well-known statements from KK-theory.

Definition 3.7 *Given an $\mathbb{F}$-vector space V with a quadratic form q we define the associated Clifford algebra*

$$Cl(V,q) = \mathcal{T}(V)/\mathcal{J}(V,q),$$

where $\mathcal{T}(V)$ is the tensor algebra of V over $\mathbb{F}$ and $\mathcal{J}(V,q)$ is the (two-sided) ideal generated by the elements $v \otimes v - q(v), v \in V$.

We note that given (V,q) and (V',q') as above then there is a canonical isomorphism $Cl(V \oplus V', q+q') \cong Cl(V,q) \otimes Cl(V',q')$.

In the following we regard $\mathbb{R}^n$ as the vector space $\mathbb{R}^n$ equipped with the standard positive definite quadratic form, and we define

$$Cl(n) = Cl(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{F}).^4$$

We then consider the $\mathbb{Z}/2$-graded vector space $C_0^\infty(\mathbb{R}^n, Cl(n))$, where the grading is induced by the grading of $Cl(n)$; and we define $E_n = L^2(\mathbb{R}^n, Cl(n))$ to be its L^2-completion, and $E^n = \mathbb{R}^n Cl(n)$.

Definition 3.8 (cf. Kasparov [13, §5], resp. [7, 4.3], and also [8, 2])

(a) *The Dirac operator $\partial_n \in \mathbf{R}_{E_n}$ is the closure of the corresponding operator defined on $C_0^\infty(\mathbb{R}^n, Cl(n))$. To present the latter let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. For a function $\sigma \in C_0^\infty(\mathbb{R}^n, Cl(n))$ of pure degree the Dirac operator is given by*

$$\partial_n \sigma(x) = (-1)^{|\sigma|} \sum_{i=1}^{n} \left(\frac{\partial \sigma}{\partial x_i}(x) \right) e_i, \quad x \in \mathbb{R}^n.$$

([8, 2.7])

(b) *Let $b_n \in \mathbf{R}_{E^n}$ be the operator given by the function*

$$b_n : \mathbb{R}^n \longrightarrow B(Cl(n)), \quad v \longmapsto L_v = \text{ left multiplication with } v.$$

([8, 2.4])

(c) *Let the Clifford operator $L_n \in \mathbf{R}_{E_n}$ be the operator defined by*

$$(L_n(\sigma))(x) = x\sigma(x), \quad x \in \mathbb{R}^n,\ \sigma \in C_0^\infty(\mathbb{R}^n, Cl(n)).$$

We then define the Bott-Dirac operator $D_n \in \mathbf{R}_{E_n}$ by $D_n = L_n + \partial_n$. ([8, 2.12])

[4] Using Kasparov's notation in [13] we have $Cl(n) = Cl_{n,0}$. This notation disagrees with the notation of many other authors!

Proposition 3.9 *We have canonical isomorphisms* $E_{n+m} = E_n \otimes E_m$ *and* $E^{n+m} = E^n \otimes E^m$. *Using these isomorphisms we have identities*

$$\partial_{n+m} = \mu(\partial_n, \partial_m), \quad b_{n+m} = \mu(b_n, b_m), \quad D_{n+m} = \mu(D_n, D_m).$$

The operators above represent well-known KK-classes, namely

$$\begin{aligned} [J(\partial_n)] &= \alpha_n \in KK(\mathbb{R}^n Cl(n), \mathbb{F}), \\ [J(b_n)] &= \beta_n \in KK(\mathbb{F}, \mathbb{R}^n Cl(n)), \\ [J(D_n)] &= \gamma_n \in KK(\mathbb{F}, \mathbb{F}). \end{aligned}$$

Theorem 3.10 (cf. [13, §5, Theorem 7], resp. [7, 4.7], resp. [18, 2.5.2]) *The Kasparov product with* α_n *respectively* β_n *induces an isomorphism*

$$\otimes_{\mathbb{R}^n Cl(n)} \alpha_n : KK(\mathbb{F}, \mathbb{R}^n Cl(n) \otimes A) \longleftrightarrow KK(\mathbb{F}, A) : \beta_n \otimes$$

and $\gamma_n = \beta_n \otimes_{\mathbb{R}^n Cl(n)} \alpha_n = 1 \in KK(\mathbb{F}, \mathbb{F})$.[5]

We use the map $\beta_n \otimes$ to identify $KK(\mathbb{F}, A)$ with $KK(\mathbb{F}, \mathbb{R}^n Cl(n) \otimes A)$. This allows us to extend the definition of $K_n(A)$ to negative indices n.

Definition 3.11 $K_{k-l}(A) = KK(\mathbb{F}, \mathbb{R}^k Cl(l) \otimes A), \quad k, l \geq 0.$

The following is another consequence of Theorem 3.10.

Corollary 3.12 *The maps*

$$\begin{aligned} \mu(b_n, .) \ &: \mathbf{R}^+_{H(A)} \to \mathbf{R}^+_{\mathbb{R}^n H(Cl(n) \otimes A)} &&, \ D \mapsto \mu(b_n, D) \\ \mu(D_n, .) \ &: \mathbf{R}^+_{H(A)} \to \mathbf{R}^+_{E_n \otimes H(A)}, &&, \ D \mapsto \mu(D_n, D) \end{aligned}$$

are weak homotopy equivalences.

Proof : We use Proposition 3.3 to identify $\pi_0(\mathbf{R}^+_{H(A)})$ with $KK(\mathbb{F}, A)$. From Proposition 3.5 it follows that $\pi_0(\mu(D_n, .))$ is given by multiplication with γ_n, which is an isomorphism by the above theorem.
To see that $\mu(D_n, .)$ is a weak homotopy equivalence let X be a compact space and consider the following commutative diagram where we used the canonical identification $H(XA) \cong XH(A)$

$$\begin{array}{ccc} [X, \mathbf{R}^+_{H(A)}] & \xrightarrow{[X, \mu(D_n, .)]} & [X, \mathbf{R}^+_{E_n \otimes H(A)}] \\ \cong \downarrow & & \cong \downarrow \\ \pi_0 C(X, \mathbf{R}^+_{H(A)}) & \xrightarrow{\pi_0 \mu(D_n, .)_*} & \pi_0 C(X, \mathbf{R}^+_{E_n \otimes H(A)}) \\ \cong \uparrow (3.4) & & \cong \uparrow (3.4) \\ \pi_0 \mathbf{R}^+_{XH(A)} & \xrightarrow{\pi_0 \mu(D_n, .)} & \pi_0 \mathbf{R}^+_{X(E_n \otimes H(A))}. \end{array}$$

[5]The operator D_n has $\mathbb{Z}/2$-graded index equal to 1 (cf. [8, 2.15]), and hence it represents the unit in $KK(\mathbb{F}, \mathbb{F})$.

As above we conclude that the bottom row is an isomorphism. Hence the map at the top of the diagram is an isomorphism. As this is true for an arbitrary compact space X the map $\mu(D_n, .)$ is a weak homotopy equivalence. The argument for $\mu(b_n, .)$ follows the same pattern (using β_n instead of γ_n). $\diamond$

4 Orthogonal spectra representing K-theory

4.1 Orthogonal spectra: Some basic definitions

Let O be the topological category whose objects are the spaces $\mathbb{R}^n$ for $n \in \mathbb{N}_0$ and whose morphisms are given by the topological groups $O(n)$ acting on $\mathbb{R}^n$. An orthogonal sequence in a topological category $\mathcal{C}$ is a continuous functor $O \to \mathcal{C}$, and the category of orthogonal sequences in $\mathcal{C}$ we denote $\mathcal{C}^O$. An orthogonal sequence in TOP_*, the category of pointed compactly generated Hausdorff spaces, consists of a sequence of pointed compactly generated Hausdorff spaces $X(0), X(1), \dots, X(n), \dots$ with a basepoint preserving (left) action of $O(n)$ on $X(n)$. A map $f : X \to Y$ in TOP_*^O is a sequence of equivariant maps $f_n : X(n) \to Y(n)$. The tensor product $X \otimes Y$ of two orthogonal sequences $X, Y \in TOP_*^O$ is the orthogonal sequence given by

$$(X \otimes Y)(n) = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} (X(p) \wedge Y(q)), \quad n \in \mathbb{N}_0 .$$

We use the unit in $O(n)$ to identify $X(p) \wedge Y(q)$ with a subspace in $(X \otimes Y)(n)$. Note that a map $f : X \otimes Y \to Z$ is determined by a sequence of pointed $O(p) \times O(q)$-equivariant maps $f_{p,q} : X(p) \wedge Y(q) \to Z(n)$. Given two maps $f : X \to X', g : Y \to Y'$ in TOP_*^O the tensor product $f \otimes g : X \otimes Y \to X' \otimes Y'$ is defined by the maps $(f \otimes g)_{p,q} = f_p \wedge g_q$. If $*$ denotes the one-point space and S^0 the zero dimensional sphere, then the unit for the tensor product is the orthogonal sequence $e = (S^0, *, *, \dots)$. Furthermore, we have a twisting map $\tau : X \otimes Y \to Y \otimes X$ given by $\tau_{p,q}(x \wedge y) = \rho_{q,p}(y \wedge x), x \in X(p), y \in Y(q)$, where $\rho_{q,p}$ is the element in the symmetric group $\Sigma_{p+q} \subset O(p+q)$ given by the block permutation

$$\begin{aligned} \rho_{q,p}(i) &= i + p \quad for \quad 1 \le i \le q \\ \rho_{q,p}(i) &= i - q \quad for \quad q < i \le p + q. \end{aligned}$$

Hovey, Shipley and Smith have shown that the category of orthogonal sequences in TOP_* with the product $\otimes$ and unit e is a symmetric monoidal category.[6]

To proceed we recall the notion of a monoid resp. a (left) module in a symmetric monoidal category. If $\mathcal{C}$ is a symmetric monoidal category with a product

[6]Hovey-Shipley-Smith actually showed the statement for TOP_*^Σ, the category of symmetric sequences in TOP_* (cf. Proposition 2.2.1 and 6.2 in [9]); the corresponding statement for TOP_*^O can be obtained analogously.

$\otimes$, a unit e and a natural twisting map $\tau : X \otimes Y \to Y \otimes X$ for $X, Y \in \mathcal{C}$, then a monoid in $\mathcal{C}$ consists of an object $R \in \mathcal{C}$, a multiplication $\mu : R \otimes R \to R$ and a unit map $\eta : e \to R$, such that the diagrams

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "Id\otimes\mu"] \arrow[d, "\mu\otimes Id"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"] & R \end{tikzcd} \qquad \begin{tikzcd} e \otimes R \arrow[r, "\eta\otimes Id"] \arrow[d] & R \otimes R \arrow[d, "\mu"] & R \otimes e \arrow[l, "Id\otimes\eta"'] \arrow[d] \\ R \arrow[r, equal] & R \arrow[r, equal] & R \end{tikzcd}$$

commute, where the two unlabeled maps correspond to the unit isomorphisms. A monoid is called commutative if $\mu = \mu \circ \tau$. If R, R' are monoids in $\mathcal{C}$ a monoid map $R \to R'$ is a map in $\mathcal{C}$ that is compatible with the structure maps.
A left R-module is an object M in $\mathcal{C}$ with an associative multiplication $\nu : R \otimes M \to M$ that respects the unit map. A right R-module is a left R^{op}-module, where R^{op} is R with the multiplication given by $\mu \circ \tau$. A map of left R-modules, of course, is a map $f : M \to N$ that is compatible with the left R-actions.

Example 4.1 The orthogonal sequence of spheres S given by $S(n) = S^n = (S^1)^{\wedge n}$ equipped with the canonical $O(n)$-actions, with the multiplication $S \otimes S \to S$ given by the canonical homeomorphisms $S^p \wedge S^q \to S^{p+q}$ and the unit map $e \to S$ which is determined by $Id_{S^0} : e(0) = S^0 \to S^0$ is a commutative monoid in TOP_*^O.

Definition 4.2 *A spectrum is a left S-module. If E is a spectrum with structure maps $\sigma_{p,q} : S^p \wedge E_q \to E_{p+q}$ then we apply the functor π_0 to the maps*

$$C_0(S^n, E_m) \longrightarrow C_0(S^{n+1}, E_{m+1}),\ f \longmapsto \sigma_{1,m} \circ (Id_{S^1} \wedge f)$$

and define the n-th homotopy group by

$$\pi_n(E) = \operatorname*{colim}_{k\to\infty} \pi_0(C_0(S^{n+k}, E_k)) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(E_k).$$

Given a commutative monoid R in TOP_*^O and two left R-modules M and N we may regard M as a right R-module and define the smash product $M \wedge_R N$ as the coequalizer

$$M \otimes R \otimes N \underset{1\otimes m}{\overset{m\otimes 1}{\rightrightarrows}} M \otimes N \longrightarrow M \wedge_R N.$$

$M \wedge_R N$ can be given the structure of a left R-module. The smash product $\wedge_R$ gives the category of left R-modules the structure of a symmetric monodial category. In particular, we obtain a symmetric monoidal structure on the category of spectra.

Definition 4.3 *A (commutative) ring spectrum is a (commutative) monoid in the (symmetric monoidal) category of spectra. A module spectrum over a ring spectrum R is a left R-module in the category of spectra.*

To actually present a ring spectrum resp. a module spectrum over a ring spectrum it is helpful to express the structure maps and compatibility conditions on the level of spaces. Doing this we obtain

Proposition 4.4

(1) *A triple (R, η, μ) consisting of a sequence of pointed spaces R_n $(n \in N_0)$ with a basepoint preserving $O(n)$-action, a sequence η of pointed equivariant maps $\eta_n : S^n \to R_n$ and a collection μ of pointed $O(m) \times O(n)$-equivariant maps $\mu_{m,n} : R_m \wedge R_n \to R_{m+n}$ defines a commutative ring spectrum if*

1. $\mu_{k+l,m} \circ (\mu_{k,l} \wedge Id_{R_m}) = \mu_{k,l+m} \circ (Id_{R_k} \wedge \mu_{l,m})$ *(associativity)*
2. $\mu_{m,n}(\eta_m \wedge \eta_n) = \eta_{m+n}$ *(compatibility)*
3. $\mu^t_{m,n} = \mu_{m,n}$ *(commutativity)*

where $\mu^t : R_m \wedge R_n \to R_{m+n}$ is defined by $\mu^t_{m,n}(e \wedge e') = \rho_{n,m}\mu_{n,m}(e' \wedge e)$ with $\rho_{n,m} \in \Sigma_{m+n}$ given by the block permutation

$$\begin{aligned} \rho_{n,m}(i) = &\quad i + m \quad \textit{for } 1 \leq i \leq n \\ \rho_{n,m}(i) = &\quad i - n \quad \textit{for } n < i \leq m + n. \end{aligned}$$

(2) *If a commutative ring spectrum R is given as above and (M, ν) is couple consisting of a sequence of pointed spaces M_n $(n \in N_0)$ with a basepoint preserving $O(n)$-action and a sequence of pointed $O(m) \times O(n)$-equivariant maps $\nu_{m,n} : R_m \wedge M_n \to M_{m+n}$ such that*

$$\nu_{k+l,m} \circ (\mu_{k,l} \wedge Id_{M_m}) = \nu_{k,l+m} \circ (Id_{R_k} \wedge \nu_{l,m}),$$

then M inherits the structure of a left R-module.

4.2 Orthogonal spectra associated to C^*-algebras

We define $H_n(A)$ by $H_n(A) = H_n \otimes A$, where $H_n = Cl(n) \otimes L^2(\mathbb{R}^n, Cl(n))$. We regard it as a right $Cl(n) \otimes A$-module where $Cl(n) \otimes A$ acts on $Cl(n) \otimes A$ by right multiplication and trivially on the factor $L^2(\mathbb{R}^n, Cl(n))$. We establish a left $O(n)$-action on $\mathbf{R}(A)^+_n = \mathbf{R}^+_{H_n(A)}$ as follows.

Let $Pin(n) \subset Cl(n)^\times$ be the subgroup generated by the elements of norm one in $\mathbb{R}^n \subset Cl(n)$ and $\mathbb{F}$ respectively. The grading on $Cl(n)$ induces a homomorphism $\alpha : Pin(n) \to \mathbb{Z}/2 = \{\pm 1\}$. The group $Pin(n)$ acts on $Cl(n)$ by so-called twisted conjugation, i.e. an element $v \in Pin(n)$ acts by $v : w \mapsto \alpha(v)vwv^{-1}$. The action has $\mathbb{R}^n$ as an invariant subspace, and on $\mathbb{R}^n$ the action is given through orthogonal transformations, so we obtain a homomorphism $\varrho_n : Pin(n) \to O(n)$. The homomorphism is a central extension of $O(n)$ by $S\mathbb{F}$, the elements of norm 1 in $\mathbb{F}$. Now let $Pin(n)$ act on $Cl(n)$ by left multiplication, and via ϱ_n on

$L^2(\mathbb{R}^n, Cl(n))$, i.e. for $v \in S^{n-1} \subset \mathbb{R}^n$ and $\sigma \in C_0^\infty(\mathbb{R}^n, Cl(n))$ the action of v on σ is given by

$$(v\sigma)(x) = Cl(\varrho_n(v) \otimes_{\mathbb{R}} Id_{\mathbb{F}})\sigma(\varrho_n(v)^{-1}x), \ x \in \mathbb{R}^n.$$

The two $Pin(n)$ actions on $Cl(n)$ and $L^2(\mathbb{R}^n, Cl(n))$ respectively induce a unitary action on H_n, and we extend this action trivially to $H_n(A)$, i.e. we let $Pin(n)$ act trivially on the factor A. We thus obtain a unitary representation $\rho_n : Pin(n) \to \mathbf{U}_{H_n(A)}$. We now consider the twisted conjugation action of $Pin(n)$ on $\mathbf{F}_{H_n(A)}$, i.e. an element $v \in Pin(n)$ acts by $v : F \mapsto \alpha(v)\rho(v)F\rho(v)^{-1}$. It factors through an effective $O(n)$-action (via ϱ_n) since the twisted conjugation action by elements in $S\mathbb{F}$ is trivial. The resulting $O(n)$-action on $\mathbf{F}_{H_n(A)}$ induces an $O(n)$-action on $\mathbf{F}^+_{H_n(A)}$ since we have

$$||UF^2U^{-1} - 1|| = ||U(F^2 - 1)U^{-1}|| = ||F^2 - 1||$$

for unitary transformations U and $F \in \mathbf{F}_{H_n(A)}$. The subspaces $\mathbf{R}_{H_n(A)} \subset \mathbf{F}_{H_n(A)}$ and $\mathbf{R}^+_{H_n(A)} \subset \mathbf{F}^+_{H_n(A)}$ are $O(n)$-invariant; thus in particular we have an $O(n)$-action on $\mathbf{R}(A)^+_n = \mathbf{R}^+_{H_n(A)}$.

Now let us assume that A is a unital $\mathbb{Z}/2$-graded C^*-algebra. We then define pointed maps $\eta(A)_n : S^n \to \mathbf{R}(A)^+_n$ by

$$\eta(A)_n(v) = \mu(L_v, D_n \otimes Id_A), \quad v \in \mathbb{R}^n.$$

These maps are $O(n)$-equivariant since the operator $D_n \otimes Id_A$ is $O(n)$-equivariant and $S(L_v) = L_{Sv}$ for all $S \in O(n)$. Note also that the maps $\eta(A)_n$ are compatible with the product map μ^+ defined in 3.6, i.e. given two unital $\mathbb{Z}/2$-graded C^*-algebras A and A' we have $\mu^+ \circ (\eta(A)_m \wedge \eta(A')_n) = \eta(A \otimes A')_{m+n}$ using the canonical isomorphism $Cl(m) \otimes A \otimes Cl(n) \otimes A' \cong Cl(m+n) \otimes A \otimes A'$. Moreover, if we are given (σ-unital) $\mathbb{Z}/2$-graded C^*-algebras B, B' and B'' and an essential $*$-homomorphism $\phi : B \otimes B' \to B''$ then the induced $*$-homomorphims $Cl(m) \otimes B \otimes Cl(n) \otimes B' \to Cl(m+n) \otimes B''$ give rise to a collection of $O(n) \times O(m)$-equivariant maps

$$\mu(\phi)_{m,n} = \mathbf{R}(\phi)^+_{m+n} \circ \mu^+ : \mathbf{R}(B)^+_m \wedge \mathbf{R}(B')^+_n \to \mathbf{R}(B \otimes B')^+_{m+n} \to \mathbf{R}(B'')^+_{m+n}.$$

Theorem 4.5

(1) *If A is a commutative $\mathbb{Z}/2$-graded unital C^*-algebra*[7] *with multiplication map $\phi_A : A \otimes A \to A$, then $\mathbf{K}(A) = (\mathbf{R}(A)^+, \eta(A), \mu(\phi_A))$ is an (orthogonal) commutative ring spectrum.*

(2) *If A is a commutative $\mathbb{Z}/2$-graded unital C^*-algebra and B a $\mathbb{Z}/2$-graded (σ-unital) C^*-algebra that is an A-module via an essential $*$-homomorphism $\phi : A \otimes B \to B$, then the pair $\mathbf{K}(B) = (\mathbf{R}(B)^+, \mu(\phi))$ makes $\mathbf{K}(B)$ a $\mathbf{K}(A)$-module spectrum.*

[7] Commutativity here means commutativity in the graded sense.

PROOF : We verify the identities presented in Proposition 4.4.

(1) The associativity condition follows from the associativity of μ^+ and associativity of ϕ_A. The compatibility condition follows from a straightforward calculation. To check commutativity we use the canonical isomorphisms

$$\begin{array}{lccc} s: & \{H_n(A) \otimes H_m(A)\} \otimes_{\phi_A} A & \xrightarrow{\cong} & H_{n+m} \otimes A \\ t: & H_n \otimes A \otimes H_m & \xrightarrow{\cong} & H_{n+m} \otimes A \end{array}$$

to obtain the formula

$$(\mu(\phi_A)_{n+m}(F, F'))^s = (F \otimes Id_{H_m})^t + (Id_{H_n} \otimes F') .$$

Computing with the right side of the equation the commutativity condition can be checked easliy.

(2) We just have to check an associativity condition. This follows from the associativity of μ^+ and the associativity and commutativity relations between the multiplication ϕ_A and the module homomorphism ϕ.

This finishes the proof of the theorem. $\diamond$

For the following now let us assume that A is a (σ-unital) $\mathbb{Z}/2$-graded C^*-algebra (and not necessarily unital). Using the canonical map $\phi : \mathbb{F}\otimes A \to A$ and Theorem 4.5 the sequence $\mathbf{R}(A)^+$ can be regarded as a $\mathbf{K}(\mathbb{F})$-module spectrum $\mathbf{K}(A)$ in a canonical way.

Theorem 4.6 *Let A be a $\mathbb{Z}/2$-graded (σ-unital) C^*-algebra. The underlying spectrum of the $\mathbf{K}(\mathbb{F})$-module spectrum $\mathbf{K}(A)$ represents $K_*(A)$, i.e. we have an isomorphism of $\mathbb{Z}$-graded abelian groups*

$$\pi_*(\mathbf{K}(A)) \cong K_*(A).$$

PROOF : By definition $\pi_n(\mathbf{K}(A)) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\mathbf{K}(A)_k)$. Let $n+k>0$ and $k>0$. Then

$$\begin{array}{rcll} \pi_{n+k}(\mathbf{R}(A)^+_k) & \cong & \pi_0(\mathbf{R}^+_{\mathbb{R}^{n+k}Cl(k)\otimes L^2(\mathbb{R}^k, Cl(k))\otimes A}) & \text{(cf. 3.3)} \\ & \cong & KK(\mathbb{F}, \mathbb{R}^{n+k}Cl(k) \otimes A) & \text{(cf. 3.11)} \\ & \cong & K_{n+k-k}(A) = K_n(A). & \end{array}$$

Moreover, if $k>0$, then the structure map $\sigma_{1,k}$ is given by

$$\sigma_{1,k} = \mu(\phi)_{1,k}(\eta(\mathbb{R})_1 \wedge Id_{\mathbf{R}(A)^+_k}).$$

The adjoint of $\sigma_{1,k}$ is

$$\begin{array}{rccl} \widetilde{\sigma}: & \mathbf{R}^+_{H_k(A)} & \longrightarrow & C_0(\mathbb{R}, \mathbf{R}^+_{H_{k+1}(A)}) \\ & F & \longmapsto & \{v \mapsto \mu(L_v, D_1) \otimes 1_{H_k(A)} + 1_{H_1} \otimes F\}, \end{array}$$

i.e. $\widetilde{\sigma}$ is given by $\mu(\mu(b_1, D_1), \cdot) = \mu(b_1, \cdot) \circ \mu(D_1, \cdot)$ composed with the canonical map $\mathbf{R}^+_{\mathbb{R}H_{k+1}(A)} \to C_0(\mathbb{R}, \mathbf{R}^+_{k+1}(A))$. The map $\mu(b_1, \cdot) \circ \mu(D_1, \cdot)$ is a weak homotopy equivalence by 3.12 and the canonical map $\mathbf{R}^+_{\mathbb{R}H_{k+1}(A)} \to C_0(\mathbb{R}, \mathbf{R}^+_{k+1}(A))$ is a homotopy equivalence by Corollary 3.4. Hence

$$\pi_{n+k}(\mathbf{R}(A)^+_k) \to \pi_{n+k+1}(\mathbf{R}(A)^+_{k+1})$$

is an isomorphism (for $n + k > 0$ and $k > 0$). The colimit stabilizes. That the group structures are compatible follows from 2.12 and 3.3 - 3.6. $\diamond$

Remark 4.7 It follows from the theory of orthogonal spectra that the category of left R-modules over a commutative orthogonal ring spectrum R has an internal Hom-functor. Hence in particular, the category of $\mathbf{K}(\mathbb{F})$-modules has an internal Hom-functor. Given two (σ-unital) C^*-algebras A and B we therefore can define the bivariant $\mathbf{K}(\mathbb{F})$-module spectrum

$$\mathbf{K}(A, B) = Hom_{\mathbf{K}(\mathbb{F})}(\mathbf{K}(A), \mathbf{K}(B)).$$

This defines a bifunctor from the category of σ-unital C^*-algebras and essential $*$-homomorphisms to the category of $\mathbf{K}(\mathbb{F})$-module spectra.

We can also define a bifunctor into the category of graded groups by associating to a pair (A, B) the graded group $\mathbf{K}_*(A, B) = [\mathbf{K}(A), \mathbf{K}(B)]_{\mathbf{K}(\mathbb{F})}$. Although the graded groups $\mathbf{K}_*(A, B)$ by construction are only functiorial with respect to essential $*$-homomorphisms, it is not hard to see that $\mathbf{K}_*(A, B)$ is indeed functorial with respect to (arbitrary) $*$-homomorphisms; moreover one can also show that $\mathbf{K}_*(A, B)$ fits into a short exact sequence

$$0 \to Ext_{\mathbf{K}(\mathbb{F})_*}(\mathbf{K}(A)_*, \mathbf{K}(B)_*) \to \mathbf{K}_*(A, B) \to Hom_{\mathbf{K}(\mathbb{F})_*}(\mathbf{K}(A)_*, \mathbf{K}(B)_*) \to 0.$$

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