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Noncommutative Homotopy theory
Exercise 11

hand in until: 16.01.2023

Exercise 1. $2x2$**-matrix tricks:** Let A be a unital C^*-algebra and a, b be in $GL(A)$, the subspace of invertible elements of A. Show that in $GL(\mathrm{Mat}_2(A))$ the elements

$$\begin{pmatrix} ab & 0 \\ 0 & 1_A \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

are homotopic.

For a C^*-algebra A we consider the subgroup

$$U^s(A) := \{U \in (A \otimes K)^u \mid 1 - U \in A \otimes K\}$$

of the unitary group of the unitalization of the stabilization of A. Conclude that the monoid $\pi_0 U^s(A)$ with operation induced by composition is commutative.

Hint: The composition is homotopic to the block sum under is isomorphism

$$U^s(A) \cong \mathrm{Hom}_{C^*\mathbf{Alg}}(C_0(S^1), A \otimes K) \ .$$

Exercise 2. A category of fibrant objects: Let $\Phi : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{C}$ be a homotopy invariant semiexact functor to a stable ∞-category. Define F to be the subcategory of $\mathbf{C}$ of homomorphisms admitting a cpc split and W as the wide subcategory of homomorphisms which are sent to equivalences by Φ. Show that $(C^*\mathbf{Alg}^{\mathrm{nu}}, W, F)$ is a category of fibrant objects.

Conclude that there is a left-exact factorization $\bar{\Phi}$.

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[rr, "\Phi"] \arrow[dr] & & \mathbf{C} \\ & C^*\mathbf{Alg}^{\mathrm{nu}}[W^{-1}] \arrow[ur, dotted, "\bar{\Phi}"] & \end{tikzcd}$$

Exercise 3. Closure by pull-backs and 2**-out-of-**3**:** Let $\mathbf{C}$ be a left-exact ∞-category with a symmetric monoidal structure which is exact in both arguments. Let D be in $\mathbf{C}$. Assume that W is a subset of morphisms in $\mathbf{C}$ which is preserved by $- \otimes D$. Show that then the smallest subcategory of $\mathbf{C}$ which contains W and is closed under pull-backs and the 2-out-of-3 property is also preserved under $- \otimes D$.

Exercise 4. Characterization of groups not using inverses: Let $(M, e, \cdot)$ be a monoid in **Set**. Show that the following are equivalent:

1. $(M, e, \cdot)$ is a group.
2. The shear map $M \times M \to M \times M$ given by $(m, m') \mapsto (m, m \cdot m')$ is a bijection.