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Noncommutative Homotopy theory
Exercise 13

hand in until: 06.02.2023

Exercise 1. $K_*(X)$ is a $K^*(X)$-module: Check that the formula for the product $K^m(X) \otimes K_n(X) \to K_{-m+n}(X)$ from the lecture defines a module structure. Show further, that for c, d in $K^*(X)$ and h in $K_*(X)$ we have $\langle c, d \cdot h\rangle = \langle c \cdot d, h\rangle$, where $\langle -, -\rangle$ denotes the KU_*-valued pairing between K-theory and K-homology.

Exercise 2. K-homology and K-theory of surfaces: Calculate $K_*(\Sigma_g)$ and $K^*(\Sigma_g)$ for a compact oriented surface Σ of genus g.
Hint: Use $KU_* \cong \mathbb{Z}[\beta, \beta^{-1}]$, homotopy invariance, suspension axiom and the Mayer-Vietoris sequences for K_* and K^*.
Describe the module structure and pairing explicitly.

Exercise 3. K-theory is invariant under inner automorphisms: Assume that A is a C^*-algebra and consider a unitary u in the multiplier algebra $M(A)$ of A. It defines an automorphism $\phi : A \to A$ by $\phi(a) := uau^*$. Show that $K(\phi) : K(A) \to K(A)$ is equivalent to the identity. Give an example of an automorphism of a C^*-algebra which does not act as identity on K-theory.
Hint: Use an exercise about matrix tricks from a previous sheet.

Exercise 4. K-theory of bounded operators is trivial: Let H be a separable ∞-dimensional Hilbert space. Show that $K(B(H)) \simeq 0$.

Hint: Choose isomorphisms $H \cong H \oplus H$ and $H \cong \bigoplus_{\mathbb{N}} H$. The first provides the second isomorphism in

$$e : B(H) \to \mathrm{Mat}_2(B(H)) \cong B(H) \ ,$$

where e is the left upper corner inclusion. The second provides the second isomorphism in

$$s : B(H) \xrightarrow{\oplus_{\mathbb{N}} \mathrm{id}} B(\bigoplus_{\mathbb{N}} H) \cong B(H) \ .$$

We now observe that

$$e \oplus s : B(H) \xrightarrow{\mathrm{diag}(\mathrm{id}, s)} \mathrm{Mat}_2(B(H)) \cong B(H)$$

is conjugated to s by some unitary in $B(H)$. It follows from Exercise 3 that

$$[e] + [s] = [s] : K(B(H)) \to K(B(H)) \ .$$

Since $K(B(H))$ is a group in **Sp** *this implies $[e] = 0$. But since $[e] = \mathrm{id}$ by stability we conclude that $K(B(H)) \simeq 0$.*

Show more generally, that $K(M^s(A)) \simeq 0$, where $M^s(A)$ is the stable multiplier algebra of A, i.e., $M^s(A) := M(A \otimes K)$.