# Seminar: Morse Theory

Fakultät für Mathematik
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(Friday 10 – 12, M102)

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## Seminar Description

The goal of this seminar is to present the basic constructions of Morse theory and some of its fundamental applications. Interesting outcomes are (the Morse-) inequalities relating the number of critical points of a function on a manifold with the Betti numbers of the manifold, a proof of the fact that a smooth manifold has the homotopy type of a CW-complex, applications to the homotopy type of path- and loop spaces of smooth manifolds, and a way to calculate the homology of a manifold using gradient flows.

Prerequisites for this seminar are the basic theory of smooth manifolds, some elements of Riemannian geometry (see, for example, [Mil63, Part II]), and some basic notions from algebraic topology.

The seminar will mainly follow the books by Milnor [Mil63] and Audin–Damian [AD14]. For some material of the seminar, the books of Hatcher [Hat02] and Nicolaescu [Nic07] will also be used.

# Schedule of Talks

## Part I. Morse functions on a manifold

### Talk 1: Morse functions (29.04.2022)

Introduce the notion of a non-degenerate critical point of a real-valued function on a smooth manifold and the notion of a Morse function [Mil63, pp. 4–5], [AD14, Section 1.1]. Show that Morse functions always exist and that every function can be approximated by a Morse function arbitrary well following [Mil63, Section 6; esp. Theorem 6.6, Corollary 6.8] and [AD14, Section 1.2]. Sard's theorem about the measure of critical values as well as Whitney's embedding theorem should be stated precisely (but used without proof).

### Talk 2: CW-complexes (06.05.2022)

Introduce the notion of a CW-complex. In particular, explain carefully the meaning of "attaching a cell". Construct the cellular chain complex of a CW-complex and show that it calculates the (singular) homology. Provide some examples of CW-complexes and compute their homologies, e.g., $\mathbb{CP}^n$ or $S^n$. The material can be found, for example, in [Hat02, pp. 137–147, 519-524].

### Talk 3: The Morse lemma (13.05.2022)

Prove the Morse lemma [Mil63, Lemma 2.2], [AD14, Section 1.3] which describes the form of a smooth function near a non-degenerate critical point. Show that a compactly supported vector field generates a flow of diffeomorphisms [Mil63, Lemma 2.4] and draw the flow near a non-degenerate critical point that arises from the gradient of a Morse function . Provide pictures where the Morse function is interpreted as a height function. Follow [Mil63, Sections 1–2] and [AD14, Sections 1.3–1.4].

### Talk 4: Topology of the sublevel sets (20.05.2022)

Prove [Mil63, Theorem 3.1] showing that the region between two level sets without intermediate critical points is a cylinder. Then prove [Mil63, Theorem 3.2] which provides the structure of the region between two level sets when there is precisely one intermediate critical point and this is non-degenerate. Show [Mil63, Theorem 3.5] which identifies the region above a level set with only non-degenerate critical points as a relative CW-complex relative to the level set. See also [AD14, Section 2.1].

### Talk 5: Some applications (27.05.2022)

Prove Reeb's Theorem [Mil63, Theorem 4.1] which states that a compact manifold with a Morse function with precisely two critical points is homeomorphic to the sphere. Then show the Morse inequalities [Mil63, Theorem 5.2] (see also [Nic07, Section 2.3]). Finally, deduce that smooth manifolds have the homotopy type of a CW-complex (following [Mil63, Theorem 3.5] and [Mil63, pp. 36–37]).

## Part II. Morse theory for the path space of a manifold

### Talk 6: Smooth paths and the energy functional (03.06.2022)

Introduce the space of piecewise smooth paths in a manifold and explain how analytical concepts are understood for this space [Mil63, Section 11]. Define the energy functional and calculate its first variation [Mil63, Theorem 12.2]. Recall the relation between energy and length and deduce that geodesics are precisely the critical points of the energy functional [Mil63, Corollary 12.3]. Follow [Mil63, Sections 11–12].

### Talk 7: The second variation of the energy functional (10.06.2022)

Calculate the second variation of the energy functional [Mil63, Theorem 13.1]. Recall the notion of a Jacobi field and its relation with the zero space of the Hessian of the energy functional [Mil63, Theorem 14.1]. Show that Jacobi fields are precisely the tangent vectors to variations of geodesics [Mil63, Lemma 14.3 and 14.4]. Follow [Mil63, Sections 13–14].

### Talk 8: The Morse index theorem (17.06.2022)

Recall the notion of conjugate points on a geodesic. Prove the Morse index theorem [Mil63, Theorem 15.1] and discuss some its consequences. Follow [Mil63, Section 15].

## Talk 9: Bounded energy path spaces (24.06.2022)

Explain the proofs of [Mil63, Theorems 16.2 and 16.3] which conclude that the energy sublevel sets of the path space between two points in a complete Riemannian manifold have the homotopy type of a finite CW-complex.

## Talk 10: Topology of the full path space (01.07.2022)

Show that the full path space of a complete Riemannian manifold has the homotopy type of a countable CW-complex [Mil63, Theorem 17.3] whose cells correspond to geodesics. As an application, discuss the homotopy type of the loop space of $S^n$ [Mil63, Corollary 17.4–17.5].

# Part III. Morse homology

## Talk 11: The Smale condition (08.07.2022)

Introduce the stable and the unstable manifold of a critical point of a gradient-like vector field associated to a Morse function [AD14, Section 2.2.a]. Explain the Smale condition and prove Smale's Theorem [AD14, Theorem 2.2.5] which states that every gradient-like vector field can be $C^1$-approximated by one satisfying the Smale condition. Follow [AD14, Section 2.2].

## Talk 12: The Morse complex (15.07.2022)

Introduce the Morse complex associated to a Morse function [AD14, Section 3.1]. In particular, explain carefully how the differential is defined and why its square vanishes. Present some examples of a Morse complex. Follow [AD14, Sections 3.1–3.2].

## Talk 13: Invariance of Morse homology (22.07.2022)

Prove [AD14, Theorem 3.4.2] stating the the homology of the Morse complex does not depend on the choice of the Morse function and the gradient-like vector field. Follow [AD14, Sec. 3.4].

### Talk 14: (to be determined) (29.07.2022)

Possible topics include: an overview of the proof of the Bott periodicity theorem (for the unitary group) [Mil63, Section 23], applications of Morse homology [AD14, Chapter 4], an introduction to degenerate critical points (see, e.g., [Lu80]), and many more.

## References

[AD14] Michèle Audin and Mihai Damian. *Morse Theory and Floer Homology.* Springer London, 2014.

[Hat02] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[Lu80] Yung Chen Lu. *Singularity theory and an introduction to catastrophe theory.* Universitext. Springer-Verlag, New York-Berlin, 1980. With a preface by Peter Hilton, Corrected reprint.

[Mil63] John Milnor. *Morse Theory. (AM-51).* Princeton University Press, dec 1963.

[Nic07] L. Nicolaescu. *An Invitation to Morse Theory.* Springer New York, 2007.