Seminar on coarse geometry

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Contents

Coarse geometry studies the large-scale properties of metric spaces and more general objects. Aspects of coarse geometry are employed in index theory, geometric group theory, algebraic K-theory, mathematical physics and many other fields. Like homology theories in topology coarse homology theories provide invariants of (bornological) coarse spaces which are often computable.

In this seminar we introduce the basic notions of coarse geometry based on the category of bornological spaces which subsumes the classical coarse spaces introduced by J. Roe as a full subcategory. We will encounter some basic coarsely invariant concepts like the set of coarse components or the Higson corona. We then introduce the notion of a coarse homology theory axiomatically and construct the example of ordinary coarse homology theory. Using euclidean cones we embed ordinary homotopy theory of finite CW-complexes into coarse homotopy theory We then discuss the concept of finite asymptotic dimension and coarse groupoids.

Here is the descriptions of the talks. The unmarked talks are core to the subject. The talks marked by $*$ could be replaced on demand by other topics in coarse geometry. Proposals are listed in Section 16.

1 Coarse structures

Introduce the notion of a coarse structure on a set, of a coarse space, and of a controlled map between coarse spaces. Describe the category **Coarse** [Bun, Sec. 2]. Show that the forgetful functor $\mathbf{Coarse} \to \mathbf{Set}$ has left- and right adjoints $X \mapsto X_{\min}$ and $X \mapsto X_{\max}$ [Bun, Lem. 2.29]. Introduce the coarse structure generated by a collection of entourages and the coarse structure induced by a map to a coarse space. Characterize the coarse spaces represented by metric spaces and discrete path metric spaces as those with a countable or a one-element generating set of entourages, respectively [Roe03, Thm. 2.55 and 2.57]. Describe push-outs, pull-backs, products and coproducts (infinitely many factors) in **Coarse** and deduce that **Coarse** is complete and cocomplete [Bun, Prop. 2.30].

2 Bornological coarse spaces

Introduce the notion of a big family on a coarse space and show that taking preimages for controlled maps preserves big families [BE, Def. 3.2]. Discuss the big family generated by a subset [BE, Ex. 4.19]. Define the notion of a bornology and explain compatibility with a coarse structure [BE, Def. 2.2], [BE, Def. 2.6]. Explain the bornology generated by a collection of subsets and analyse when it is compatible. Introduce the notion of a proper map and the category $\mathbf{BC}$ of bornological coarse spaces [BE, Def. 2.11]. Show that metric spaces present bornological coarse spaces. Explain the symmetric monoidal structure $\otimes$ in $\mathbf{BC}$ [BE, Ex. 2.32]. Describe coproducts and products in $\mathbf{BC}$ and also discuss the free union. Show by example, that not all colimits exist in $\mathbf{BC}$, see [Hei19, Sec. 3].

3 Coarse equivalences

Introduce the notion closeness of maps and of a coarse equivalence [BE, Def. 3.13-14]. Show that $\mathbb{Z}^n \to \mathbb{R}^n$ is a coarse equivalence. Define the notions of V-dense and V-separated subsets [BE, Def. 8.15]. Show the existence of maximal V-separated subsets and that they are V^2-dense, or even V-dense if V is symmetric [BE, Lem. 819]. Show that a bornological coarse space is always coarsely equivalent to any V-discrete subset with the induced structures. Define the notions of V-capacity and V-entropy [Roe03, Def. 3.1]. Discuss [Roe03, 3.3-3.7] and finally define the notion of bounded geometry.

4 Inverting coarse equivalences *

Recall the notion of a localization of a category at a set of morphisms. Define the quotient functor $q : \mathbf{BC} \to \mathbf{BC}_h$ which identifies close maps and show that it is universal among such functors (if possible every in the context of ∞-categories). Introduce the notion of a coarsely excisive decomposition (Y, Z) of a bornological coarse space [BE, Def. 3.41] and show that $X \cong Y \sqcup_{Y \cap Z} Z$ in $\mathbf{BC}$ and also in $\mathbf{BC}_h$. Describe coequalizers in $\mathbf{BC}_h$. For an action of G on X define the bornological coarse space $X//G$ and show that $X//G$ represents $\mathtt{colim}_{BG} X$ in $\mathbf{BC}_h$ [Bun25, Sec. 2].

5 G-Bornological coarse spaces

Introduce G-bornological coarse spaces and explain the difference to bornological coarse spaces with a G-action [BEKW, Sec. 2.1]. Show that metric spaces with isometric actions represent G-bornological spaces and explain why isometric is important and e.g. Lipschitz is too general. Define the canonical structure $G_{\mathrm{can,min}}$ on a group and show that it is a G-bornological coarse space [BEKW, Ex. 2.4]. Show further that this structure is equivalent to the metric structure given by the word metric associated to finite generating set if G is finitely generated. What happens if G is not finitely generated. Show that $G_{\mathrm{can,min}} \cong G_{\mathrm{min,min}}//G$ [Bun25, Ex. 2.7]. Show the Milnor-Švarc theorem stating that the universal covering $\tilde{X}$ of a compact locally contractible path-metric space X is again a proper path-metric space which is coarsely equivalent $\pi_1(X,x)_{\mathrm{can,min}}$, [Roe03, Thm. 1.18].

6 Coarse homotopies

Introduce the notion of a coarse cylinder and of a coarse homotopy [Bun, Sec. 6]. Show that (orientation-preserving) rotations of $\mathbb{R}^n$ are not close to the identity but coarsely homotopic to the identity [Bun, Lem. 6.10]. Introduce the euclidean cone functor on the category of closed subsets of S^n (for varying n) and Lipschitz maps which sends a subset $X \subseteq S^n$ to the bornological coarse space $[0,\infty)X \subseteq \mathbb{R}^{n+1}$ with the induced structures. In particular check functoriality. Show that this functor sends homotopies to coarse homotopies and closed decompositions to coarsely excisive decompositions (the task should be understood as exercises to be solved on your own).

7 The Higson corona

Recall Gelfand duality between compact Hausdorff spaces $\mathbf{CHaus}^{\mathrm{op}}$ and unital commutative C^*-algebras $\mathbf{CommC^*Alg}$. For a bornological coarse space introduce the algebra of functions with vanishing variation at ∞ and define the Higson compactification as the Gelfand dual of this algebra and characterize the Higson corona similarly [Roe03, Sec. 2.3] (only considers a special case), [Bun23, Ex. 4.6], [BL24, Def. 3.2] in the case where the big family is the bornology. Explain [Roe03, Prop. 2.47]. Show that forming the Higson corona is a functor $\partial^H : \mathbf{BC} \to \mathbf{CHaus}$. Show that ∂^H identifies close maps. Construct a canonical map $\partial^H \mathbb{R}^{n+1} \to S^n$ and show that it is surjective, but not injective (exercise part).

8 Properties of functors on BC

For functors $E : \mathbf{BC} \to \mathbf{M}$ discuss the notions of coarse invariance, vanishing on flasques and u-continuity. Follow the discussion in [Bun, Sec. 4]. Explain coarse invariance, show [Bun, Lemma 4.2] and provide examples of coarsely invariant functors. Introduce the notion of flasque (also that of flasque in the generalized sense) bornological coarse spaces. Provide examples and show [Bun, Lemma 4.12]. Finally introduce u-continuous functors and provide examples. Show that $X \mapsto \partial^H X$ from $\mathbf{BC}$ to $\mathbf{CHaus}$ (the Higson corona functor) is u-continuous (exercise part).

9 Pairs, δ-functors and coarse homology theories

Introduce the category $\mathbf{BC}^2$ of pairs $(X, \mathcal{Y})$ of bornological coarse spaces and a big family. Explain the construction of a functor ∂E and the transformation $\partial E \to uE$. Define the notion of a complementary pair and introduce excisiveness of a functor $\mathbf{BC}^2 \to \mathbf{M}$ [Bun, Def. 4.21, 4.24]. Introduce the notion of a coarse δ-functor and state the definition of a coarse homology theory (H_*, ∂) [Bun, Def. 4.29, 4.30]. If you understand ∞-categories, then discuss the advantages of an ∞-categorical definition [Bun, Rem 4.26]. Use the axioms to show that $H_*(\mathbb{R}^n) \cong H_*(\mathbb{Z}^n) \cong H_{*-n}(*)$ for any coarse homology theory [Bun, Ex. 4.31].

10 Coarse ordinary homology

Construct the coarse ordinary homology functor. Provide the construction of the respective chain complex functor $C\mathcal{X} : \mathbf{BC} \to \mathbf{Ch}$ [BE, Sec. 6.3], [Bun, Sec. 5]. Define the boundary map δ for $H\mathcal{X}$ as in [Bun, Def. 5.4]. Verify that $(H\mathcal{X}, \delta)$ satisfies the axioms of a coarse homology theory. Calculate the value on the point. Show that it is strongly additive, i.e., sends free unions to products. The details for all this can be found in [Bun, Sec. 5].

11 Further properties and calculations in coarse homology

Show that coarse homology theories send coarsely homotopic maps to equal maps [Bun, Cor. 6.8]. Calculate the effect of an orthogonal rotation on $\mathbb{R}^n$ in coarse homology. Calculate the ordinary coarse homology of the subset $\{n^2 \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ with the induced structures [Bun, Ex. 4.40]. Compare the results with the values on $\mathbb{N}_{\mathrm{disc}}$ and $\mathbb{N}$. Relate the coarse homology of the euclidean cone on a finite CW-complex X (it can be embedded into some S^n) and the usual homology of X known from algebraic topology (exercise part, use that the composition of the cone functor with coarse homology is excisive and homotopy invariant).

12 Finite asymptotic dimension*

Introduce the concept of finite asymptotic dimension [Roe03, Sec. 9.1]. Give some examples of spaces with known asymptotic dimension. Show a version of [Roe03, Thm. 9.9] about equivalent characterizations of finite asymptotic dimension in the context of bornological coarse spaces (details on this topic from the script of the 2024/25 lecture course will be supplied on demand). Show the basic properties [Roe03, Prop. 9.10-11] of the asymptotic dimension for products and taking subspaces.

13 Finite asymptotic dimension II*

Introduce uniform finite asymptotic dimension for families. State and show [Roe03, Prop. 9.15] on finite asymptotic dimension of certain infinite unions. Further show [Roe03, Prop. 9.16] and deduce [Roe03, Prop. 9.17-18] for finite asymptotic dimensions of groups. Deduce that an extension of groups of finite asymptotic dimension has again finite asymptotic dimension.

14 Stone-Čech compatifications*

This talk is independent of coarse geometry and provides preparatory material. Introduce the Stone-Čech compactification βX of a set and describe its clopen subsets and points in the boundary [Roe03, Sec. 7.4]. Describe the clopen subsets in $\beta X \times \beta Y$ in terms of decomposable subsets [Roe03, Sec. 10.3]. Show [Roe03, Sec. Prop.15] and discuss [Roe03, Sec. Prop.16].

15 The coarse translation groupoid*

Recall the notion of a uniformly discrete coarse space. Following [Roe03, Sec. 10.3] introduce the translation groupoid $G(X)$ of a coarse space and show [Roe03, Sec. 10.10]. Discuss [Roe03, Sec. 10.13].

16 Further topics

1. ∞-category view on coarse homology theories and the universal coarse homology theory (needs knowledge of ∞-categories) [BE].

2. Roe algebras and coarse K-homology (needs knowledge in functional analysis) [BE, Sec. 8], [Roe03, Sec. 4].

3. Finite asymptotic dimension, exactness, and Yu's property A [Roe03, Sec. 11].

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