Vorlesung: Introduction to homotopy theory

Ulrich Bunke

Contents

1 Covering theory

1.1 Paths and path components

We let $I := [0,1] \subset \mathbb{R}$ denote the standard interval with its induced topology.

Let Y be a topological space and $y_0, y_1 \in Y$ be two points. A **path** in Y from y_0 to y_1 is a map $\gamma : I \to Y$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$.

We consider three points $y_0, y_1, y_2 \in Y$. If γ is a path from y_0 to y_1 and μ is a path from y_1 to y_2, then we can define a new path $\mu \circ \gamma$ from y_0 to y_1 called the **concatenation** of γ and μ. It is given by

$$(\mu \circ \gamma)(t) := \begin{cases} \gamma(2t) & t \in [0, 1/2) \\ \mu(2t-1) & t \in [1/2, 1] \end{cases} \quad .$$

We say that y_0 and y_1 belong to the same **path component** of Y if there exists a path from y_0 to y_1. The relation between points y_0 and y_1 of Y

$$y_0 \text{ and } y_1 \text{ belong to the same path component of } Y$$

is an equivalence relation on Y.

Problem 1.1. *Show this assertion.*

The equivalence classes with respect to this equivalence relation are called the **path components** of Y. The set of path components will be denoted by $\pi_0(Y)$. The symbol $[y]$ denotes the path component of Y which contains y. A space Y is called **path-connected**, if $\pi_0(Y)$ has at most one element.

Example 1.2. The standard interval I is path-connected.

Example 1.3. The space $\mathbb{R}^n$ is path connected.

Example 1.4. More generally, a manifold M is path connected if and only if the following cohomological condition is satisfied:

$$\dim_{\mathbb{R}} H^0_{dR}(M) \leq 1 \ .$$

Problem 1.5. *Show this assertion.*

Problem 1.6. *Show that for general topological spaces the condition of being path connected is strictly stronger than the condition of being connected.*

Example 1.7. The subset $\mathbb{Q} \subset \mathbb{R}$ with the induced topology is not path-connected. Every path in $\mathbb{Q}$ is constant. In fact, $\mathbb{Q} \ni q \mapsto [q] \in \pi_0(\mathbb{Q})$ is a bijection.

Example 1.8. Let $p \in \mathbb{N}$ be a prime. Then the space of p-adic numbers $\mathbb{Z}_p$ is not path-connected. Again, every path in $\mathbb{Z}_p$ is constant and $\mathbb{Z}_p \ni x \mapsto [x] \in \pi_0(\mathbb{Z}_p)$ is a bijection.

Example 1.9. If G is a topological group, then $\pi_0(G)$ is a group with operations defined on representatives by

$$[g][h] := [gh] \ , \quad g, h \in G \ .$$

For example,

$$\pi_0(GL(n, \mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z} \ ,$$

where the isomorphism is given by the sign of the determinant.

Problem 1.10. *Show this assertion*

1.2 Lifting properties

We consider the following diagram of topological spaces:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \ .$$

We understand the bold part as given data.

If the dotted arrow exists for all choices of horizontal arrows, then we say that f has the **right lifting property (RLP)** with respect to i or, equivalently, i has the **left lifting property (RLP)** with respect to f. We add the adjective **unique** if the dotted arrow is unique and abbreviate this by ULLP or URRP, respectively.

We now consider the lifting of paths.

Definition 1.11. *A map of spaces $f : X \to Y$ has the* ***(unique) path lifting property****, if it has the RLP (URLP) with respect to the inclusion of the beginning $\{0\} \to I$ of the interval.*

We spell this out. We consider the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "f"] \\ I \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & Y \end{tikzcd} \ .$$

In this situation the path $\tilde{\gamma}$ is called a **lift** of γ with **beginning** in $x_0 \in X$.

The map $f : X \to Y$ has the (unique) path lifting property, if for every datum (γ, x_0) as above a (unique) lift $\tilde{\gamma}$ exists.

Example 1.12. Let $X \to Y$ be a real vector bundle over a smooth manifold Y. Then $X \to Y$ has the path lifting property.

If the path is smooth, then a lift can be found using differential geometry. Indeed, one can choose a connection and define $\tilde{\gamma}$ using the parallel transport along γ.

For continuous path we use Lemma 1.14.

Example 1.13. If $f : X \to Y$ is a smooth proper submersion between manifolds, then f has the path lifting property.

For smooth paths the argument is similar as in Example 1.12 using a (non-linear) connection. In order to lift continuous paths we observe that $f : X \to Y$ is a locally trivial fibre bundle and apply Lemma 1.14.

The condition that f is proper can not be dropped. As a counterexample let $f : (-1, 1/2) \to (-1, 2)$ be the inclusion. The path $\gamma : I \to (-1, 2)$ given by the obvious inclusion has no lift with beginning in 0.

Lemma 1.14. *A locally trivial fibre bundle $f : X \to Y$ has the path lifting property.*

Proof. Let a diagram

$$\begin{tikzcd} \{0\} \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "f"] \\ I \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & Y \end{tikzcd}$$

be given. Then the image γ can be covered by a finite number of open subsets over which the bundle is trivial. We can write the path γ as a multiple concatenation of paths which are contained in such open subsets. Since we can concatenate the lifts as well, we can reduce the problem to the case of a trivial bundle. We assume that $X = Y \times F$ and write $x_0 = (\gamma(0), f_0)$. Then can define a lift by $t \mapsto \tilde{\gamma}(t) := (\gamma(t), f_0)$. □

If ϕ is a path in F starting in f_0, then we can consider the lift $t \mapsto (\gamma(t), \phi(t))$. This accounts for the non-uniqueness of the lift.

Definition 1.15. *A locally trivial fibre bundle with discrete fibres is called a **covering**.*

Example 1.16. Let X be a topological space an G be a group which acts freely and properly on X. Then the projection $X \to X/G$ is a covering. More concrete examples of coverings are

1. $\mathbb{R}^n \to T^n \cong \mathbb{R}^n/\mathbb{Z}^n$
2. $S^3 \to L(p,q) \cong S^2/\mathbb{Z}^p$ ($L(p,q)$ is the lense space)
3. $S^n \to \mathbb{RP}^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$

Lemma 1.17. *If $f : X \to Y$ is a covering, then f has the unique path lifting property.*

Proof.

$$\begin{tikzcd} \{0\} \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "f"] \\ I \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & Y \end{tikzcd}$$

be given. As in the proof of Lemma 1.14 we can reduce to the case where the covering is trivial $X = Y \times F \to Y$ for a discrete space F. The only way to lift γ is as $\tilde{\gamma}(t) := (\gamma(t), f_0)$. □

Remark 1.18. Note that the argument works if the fibre of f just has the property that every path in it is necessarily constant. An example is $Y \times \mathbb{Z}_p \to Y$. Hence the unique path lifting property does not imply that our map is a covering.

For a space A we consider URLP for the inclusion $i_0 : A \to I \times A$ induced by $0 \in I$. A map $f : X \to Y$ which has the URLP with respect to this map is said to have the **unique homotopy lifting property for** A.

Lemma 1.19. *A covering has the unique homotopy lifting property for all spaces.*

Proof. We consider a diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{h}_0"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ I \times A \arrow[r, "h"] \arrow[ur, dotted, "\tilde{h}"] & Y \end{tikzcd} \quad .$$

For every $a \in A$ we can define $\tilde{h}_{|I \times \{a\}}$ using the unique path lifting property. It remains to check that $\tilde{h}$ is continuous. This can be done locally. Using concatenation we reduce to the case that $f : Y \times F \to Y$ is the projection. But then $\tilde{h}(t,a) = (h(t,a), \mathtt{pr}_F \tilde{h}_0(a))$. This map is obviously continuous. □

1.3 The fundamental groupoid

We consider a space X and a pair of points $x_0, x_1 \in X$. By $P_{x_0,x_1}(X)$ we denote the set of all paths from x_0 to x_1. We say that two paths $\gamma_0, \gamma_1 \in P_{x_0,x_1}(X)$ are **homotopic** if there exists a homotopy h filling the diagram

$$\begin{tikzcd} \{0,1\} \times I \cup I \times \{0,1\} \arrow[r, "(\gamma_0 \sqcup \gamma_1) \cup (x_0 \sqcup x_1)"] \arrow[d] & X \\ I \times I \arrow[ur, "h"] & \end{tikzcd} \quad .$$

Here the right vertical map is the inclusion of the boundary of the square $\partial(I \times I) \to I \times I$ and the upper arrow fixes the restriction of h at the boundary as indicated. In other words, we have

$$h(i,t) = \gamma_i(t) \ , \quad h(s,i) = x_i$$

for $i \in \{0,1\}$ and $s,t \in I$.

Being homotopic is an equivalence relation on $P_{x_0,x_1}(X)$.

Problem 1.20. *Show this assertion.*

We let $\Pi_{x_0,x_1}(X)$ denote the set of equivalence classes $[\gamma]$ of paths $\gamma \in P_{x_0,x_1}(X)$.

Given paths $\gamma \in P_{x_0,x_1}(X)$ and $\mu \in P_{x_1,x_2}(X)$ we have the concatenation $\mu \circ \gamma \in P_{x_0,x_2}(X)$. This path is obtained by running first through γ and then through μ, in double speed. The concatenation induces a well-defined operation

$$\circ : \Pi_{x_1,x_2} \times \Pi_{x_0,x_1} \to \Pi_{x_0,x_2} \ .$$

Problem 1.21. *Show this assertion.*

Let $\nu \in P_{x_2,x_3}(X)$ be a third path. The concatenation is not associative since in general

$$\nu \circ (\mu \circ \gamma) \neq (\nu \circ \mu) \circ \gamma \ .$$

Indeed, on the left γ is run through in fourfold speed, while on the right it is run through in double speed. But the parametrizations of the sides are homotopic. Hence concatenation is associative on the level of homotopy classes:

$$[\nu \circ (\mu \circ \gamma)] = [(\nu \circ \mu) \circ \gamma] \ .$$

Problem 1.22. *Show this assertion.*

If $\gamma^{-1} \in P_{x_1,x_0}(X)$ denotes the path γ run in the opposite direction, and $c_x \in P_x(X)$ denotes the constant path at x, then we have the relations

$$[\gamma^{-1}] \circ [\gamma] = [c_{x_0}] \ , \quad [\gamma] \circ [\gamma^{-1}] = [c_{x_1}] \ , \quad [\gamma] \circ [c_{x_0}] = [\gamma] = [c_{x_1}] \circ [\gamma] \ .$$

Problem 1.23. *Verify these equalities.*

Recall that a **groupoid** is a small category in which all morphisms are invertible. The upshot of the discussion above is that we have a groupoid, denoted by $\Pi(X)$, whose objects are the points of X, with set morphisms $\Pi_{x_0,x_1}(X)$ from x_0 to x_1, and with the composition and identities given by concatenation of paths and the classes of constant paths $[c_x]$.

Problem 1.24. *Show this assertion.*

Definition 1.25. *The groupoid $\Pi(X)$ is called the **fundamental groupoid** of X.*

We have a category **Groupoids** whose objects are groupoids, and whose morphisms are functors between groupoids. A map $f : X \to Y$ between topological spaces induces a morphism of groupoids as follows:

$$\Pi(f) : \Pi(X) \to \Pi(Y) \ , \quad \Pi(f)(x) := f(x) \ , \quad \Pi(f)([\gamma]) := [f \circ \gamma] \ .$$

Problem 1.26. *Show this assertion.*

Let **Top** denote the category of topological spaces and continuous maps. We thus get a functor

$$\Pi : \mathbf{Top} \to \mathbf{Groupoids} \ , \quad X \mapsto \Pi(X) \ .$$

Problem 1.27. *Show this assertion.*

Remark 1.28. The set of isomorphism classes of $\Pi(X)$ can naturally be identified with $\pi_0(X)$. We thus get a functor

$$\pi_0 : \mathbf{Top} \to \mathbf{Set} \ .$$

The category of groupoids is actually a 2-**category**. Namely, if C, D are groupoids, then the set $\mathbf{Fun}(C, D)$ is the set of objects of the category of functors from C to D. The morphisms in this category are natural isomorphism between functors. Besides the notion of isomorphism of groupoids (a categorical concept of the 1-category of groupoids) we have the 2-categorical concept of an equivalence between groupoids. A functor $f : C \to D$ is an **equivalence**, if there exists a functor $g : D \to C$ such that $f \circ g$ is isomorphic to $\mathtt{id}_D$ in $\mathbf{Fun}(D, D)$, and $g \circ f$ is isomorphic to $\mathtt{id}_C$ in $\mathbf{Fun}(C, C)$.

Let $f_0, f_1 : X \to Y$ be maps and $H : I \times X \to Y$ be a homotopy, i.e. a map fitting in the diagram

$$\begin{tikzcd} \{0,1\} \times X \arrow[r, "f_0 \sqcup f_1"] \arrow[d] & X \arrow[d] \\ I \times X \arrow[r, "H"] & Y \end{tikzcd} .$$

Then H induces a natural isomorphism $\Pi(H) : \Pi(f_0) \to \Pi(f_1)$ by

$$\Pi(H)(x) := [H_{|I \times \{x\}}] : \Pi(f_0)(x) \to \Pi(f_1)(x) \ , \quad x \in \Pi(X) \ .$$

If $f : X \to X$ is homotopic to the identity, then $\Pi(f)$ is isomorphic to the identity. Consequently, Π maps homotopy equivalences between topological spaces to equivalences between groupoids.

Proposition 1.29. *Fill in the missing arguments.*

Remark 1.30. In fact, one can refine the category **Top** to a 2-category where the objects of the category of morphisms $X \to Y$ are continuous maps, and morphisms in this category are homotopies. Then Π refines to a functor between 2-categories.

1.4 The fundamental group

Let X be a space and $x \in X$. We call the pair (X, x) a **pointed space**.

Definition 1.31. *The* ***fundamental group*** *of (X, x) is the group of automorphisms of x in $\Pi(X)$, i.e.*

$$\pi_1(X, x) := \Pi_{x,x}(X) \ .$$

A map of pointed spaces $f : (X, x) \to (Y, y)$ is a map of spaces such that $f(x) = y$. It induces a homomorphism of groups

$$\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$$

by restriction of Π. Let $\mathbf{Top}_*$ denote the category of pointed spaces and maps between pointed spaces. We thus get a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Groups} \ , \quad (X, x) \mapsto \pi_1(X, x) \ .$$

Problem 1.32. *Let $i : X_x \subset X$ be the inclusion of the path component of x. Show that*

$$\pi_1(i) : \pi_1(X_x, x) \to \pi_1(X, x)$$

is an isomorphism.

The conjugation by a class $[\mu] \in \Pi_{x_0,x_1}(X)$ induces an isomorphism of groups

$$\pi_1(X, x_0) \to \pi_1(X, x_1) \ , \quad [\gamma] \mapsto [\mu][\gamma][\mu^{-1}] \ .$$

Therefore, if x_0 and x_1 belong to the same path component of X, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic as groups, but the isomorphism may depend on the choice of a homotopy class paths from x_0 to x_1. The isomorphism is unqiue up to an inner automorphism.

Corollary 1.33. *If X is path-connected, then the group $\pi_1(X, x)$ is independent of the choice of the base point x up to an inner automorphism.*

A path connected space X is called **simply connected**, if $\pi_1(X, x) = 0$ for one, and hence for every choice of a base point $x \in X$.

Remark 1.34. In terms of the fundamental groupoid one can say that a space X is connected iff $\Pi(X)$ is empty or has only one isomorphism class of objects. In the latter case we also say that $\Pi(X)$ is connected. Furthermore, the space X is simply connected, iff $\Pi(X)$ is equivalent to the empty groupid or the final groupoid.

A space is called **contractible**, if there exists a point $x \in X$ and a homotopy of maps $X \to X$ from $\mathtt{id}_X$ to the constant map with value x.

Problem 1.35. *Show that a contractible space is simply connected.*

Example 1.36. The space $\mathbb{R}^n$ is contractible and hence simply connected.

Example 1.37. We consider the manifold S^1. We observe that any class $[\gamma]$ can be represented by a smooth path γ, and that any two smooth representatives are homotopic via a smooth homotopy. This is shown by smoothing the continuous paths or homotopies.

Let $\alpha \in \Omega^1(S^1)$ be the normalized volume form and $[\alpha] \in H^1_{dR}(S^1)$ be its de Rham cohomology class. Since de Rham cohomology is homotopy invariant, the following homomorphism is well-defined:

$$\pi_1(S^1, 1) \to \mathbb{Z} \ , \quad [\gamma] \mapsto \int_{S^1} \gamma^*[\alpha] \ .$$

The integral yields an integer, since it computes the mapping degree of $\gamma : S^1 \to S^1$. This map is actually an isomorphism. We will show this fact later.

1.5 Coverings and representations of the fundamental groupoid

For a space Y we let $\mathbf{Cov}(Y)$ denote the **category of coverings** of Y. The objects of $\mathbf{Cov}(Y)$ are coverings $(f : X \to Y)$. A morphism of coverings

$$(f : X \to Y) \to (f' : X' \to Y)$$

is given by a map $g : X \to X'$ which preserves fibres, i.e. the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

commutes. The goal of this subsection is to construct a functor

$$\Phi : \mathbf{Cov}(Y) \to \mathbf{Fun}(\Pi(Y), \mathbf{Set})$$

and to analyze when it is an equivalence of categories. We will call $\mathbf{Fun}(\Pi(Y), \mathbf{Set})$ the category of **representations** of the groupoid $\Pi(Y)$.

We start with the construction of the functor Φ. Let $f : X \to Y$ be a covering. Then $\Phi(f) : \Pi(Y) \to \mathbf{Set}$ is the representation which sends the object $y \in \Pi(Y)$ (i.e. a point $y \in Y$) to the set $\Phi(f)(y) := f^{-1}(y)$, i.e. the fibre of f at y. Let now $[\gamma] \in \Pi_{y_0,y_1}(Y)$ be a morphism. Then $\Pi([\gamma]) : f^{-1}(y_0) \to f^{-1}(y_1)$ sends the point $x_0 \in f^{-1}(y_0)$ to the endpoint of the unique lift of γ with beginning in x_0. Since a covering has the unique homotopy lifting property, this endpoint is well-defined independently of the choice of the representative of the homotopy class $[\gamma]$.

Problem 1.38. *Check that $\Phi(f)$ is a representation. Further check, that Φ is a functor.*

Under certain conditions we can reconstruct the covering $f : X \to Y$. We assume that Y is **locally simply connected**. This means, that every point $y \in Y$ has a basis of simply-connected neighborhoods. Under this assumption we construct a functor

$$\Psi : \mathbf{Fun}(\Pi(Y), \mathbf{Set}) \to \mathbf{Cov}(Y) \ .$$

Let $C \in \mathbf{Fun}(\Pi(Y), \mathbf{Set})$ be a representation of $\Pi(Y)$. Then we define the set

$$X := \bigsqcup_{y \in Y} C(y) \ .$$

This is the candidate for the total space of the covering. It has a natural map $f : X \to Y$. It remains to define an appropriate topology on X so that this map becomes a covering.

For $y \in Y$ let U_y be a simply-connected neighborhood. For every $y' \in U_y$ we have a canonical isomorphism $C(y') \to C(y)$ induced by a the unique homotopy class paths from y' to y in U_y. We get a bijection

$$f^{-1}(U_y) \cong U_y \times C(y) \ .$$

We equip X with the minimal topology such that the bijections $f^{-1}(U_y) \to U_y \times C(y)$ are continuous for all choices of $y \in Y$ and $U_y \subseteq Y$ as above. In order to show that $X \to Y$ is a covering we must show that these bijections are homeomorphisms. They are continuous by definition. We must check that the inverses are continuous as well. To this end we show that the transition maps are given by a locally constant cocycle. For two points y_0, y_1 we get a transition map

$$(U_{y_0} \cap U_{y_1}) \times C(y_0) \cong f^{-1}(U_{y_0} \cap U_{y_1}) \cong (U_{y_0} \cap U_{y_1}) \times C(y_1)$$

of the form

$$(y, a) \mapsto (y, \phi(y, a)) \ .$$

Observe that $\phi = C([\mu^{-1}] \circ [\gamma])$, where $[\gamma]$ is the unique homotopy class of paths from y_0 to y in U_{y_0} and $[\mu]$ is the unique homotopy class of paths from y_1 to y in U_{y_1}. It is easy to check that this homotopy class $[\mu^{-1}] \circ [\gamma] \in \Pi(Y)_{y_0, y_1}$ is locally constant in y.

Problem 1.39. *Give the details.*

Consequently we get a locally trivial fibre bundle with discrete fibres.

We let $\Psi(C)$ be the covering constructed above.

Problem 1.40. *Extend the construction of Ψ to morphisms.*

Problem 1.41. *Show that Φ and Ψ are inverse to each other equivalences of categories.*

The upshot of this discussion is:

Proposition 1.42. *If Y is a locally simply-connected space, then Φ induces an equivalence of categories*

$$\mathbf{Cov}(Y) \simeq \mathbf{Fun}(\Pi(Y), \mathbf{Set}) .$$

Problem 1.43. *Show that a map $f : Y' \to Y$ induces a functor*

$$\mathtt{Cov}(f) : \mathbf{Cov}(Y) \to \mathbf{Cov}(Y') , \quad (X \to Y) \mapsto (Y' \times_Y X \to Y') .$$

If f is a homotopy equivalence between locally simply-connected spaces, then $\mathtt{Cov}(f)$ is an equivalence.

Example 1.44. The identity map $Y \to Y$ is a covering. The corresponding functor $\Pi(Y) \to \mathbf{Set}$ is the constant functor which sends every point of Y to a point and every morphism to the identity.

More generally, let F be a set and consider the trivial covering $Y \times F \to Y$. The corresponding functor $\Pi(Y) \to \mathbf{Set}$ is the constant functor which sends every point of Y to F and every morphism to the identity of F.

Assume that Y is locally simply-connected. Given a representation $C \in \mathbf{Fun}(\Pi(Y), \mathbf{Set})$ we consider a covering $\Psi(C) \in \mathbf{Cov}(Y)$. It is then a natural problem to describe the fundamental groupoid $\Pi(\Psi(C))$ in terms of C.

To this end we introduce the **transport category** $T(C)$ of a representation $C : G \to \mathbf{Set}$ of a groupoid G. The set of objects of $T(C)$ is given by $\bigsqcup_{g \in G} C(g)$. Let $g_0, g_1 \in G$ and $x_i \in C(g_i)$ for $i = 0, 1$. Then x_0, x_1 are objects of $T(C)$. The sets of morphisms from x_0 to x_1 is defined by

$$\mathtt{Hom}_{T(C)}(x_0, x_1) := \{\phi \in \mathtt{Hom}_G(g_0, g_1) \mid C(\phi)(x_0) = x_1\} .$$

The composition is induced by the composition in G.

Problem 1.45. *Show that $T(C)$ is a well-defined groupoid.*

Proposition 1.46. *Assume that Y is locally simply-connected. For $C \in \mathbf{Fun}(\Pi(Y), \mathbf{Set})$ there is a natural isomorphism of groupoids*

$$\Pi(\Psi(C)) \cong T(C)) .$$

Proof. By construction of Ψ we have a canonical bijection between the sets of objects of $T(C)$ and the points of $\Psi(C)$. We extend this bijection to a functor $A : T(C) \to \Pi(\Psi(C))$. So we must define the action of this functor on morphisms. For $i = 0, 1$ we consider points $y_i \in Y$ and $x_i \in C(y_i) \subseteq X$. Let $[\gamma] : x_0 \to x_1$ be a morphism in $T(C)$. Then $[\gamma] \in \Pi_{y_0,y_1}(Y)$ is such that $C([\gamma])(x_0) = x_1$. Since Φ inverts Ψ we see that unique lift $\tilde{\gamma}$ of γ to $\Psi(C)$ with beginning in x_0 has the end-point x_1. Therefore we have $[\tilde{\gamma}] \in \Pi_{x_0,x_1}(X)$. We set $A([\gamma]) := [\tilde{\gamma}]$.

This construction is compatible with the composition.

The inverse functor $B : \Pi(\Psi(C)) \to T(C)$ maps $[\tilde{\gamma}] \in \Pi_{x_0,x_1}(X)$ to $[\gamma] \in \Pi_{y_0,y_1}(Y)$. To this end we observe that the path γ obtained by projecting $\tilde{\gamma}$ to Y really belongs to $\mathtt{Hom}_{T(C)}(x_0, x_1)$.

Problem 1.47. *Verify this.*

$\square$

1.6 Specialization to the fundamental group

If G is a groupoid, then we write $\pi_0(G)$ for the set of isomorphism classes of G. Note that an equivalence $H \xrightarrow{\sim} G$ of groupoids induces a bijection $\pi_0(H) \xrightarrow{\cong} \pi_0(G)$ between the sets of their isomorphism classes. A groupoid G is called **connected** if for every pair of objects $g, g' \in G$ the set $\mathtt{Hom}_G(g, g')$ is not empty. Equivalently, the set $\pi_0(G)$ has at most one element.

Example 1.48. The fundamental groupoid $\Pi(Y)$ of a space Y is connected if and only if the space Y is path-connected. More generally we have a canonical bijection $\pi_0(Y) \cong \pi_0(\Pi(Y))$.

We assume that G is a groupoid and $g \in G$ is an object. Then we can consider the group $\mathtt{Aut}_G(g)$ as a groupoid with one object g.

Lemma 1.49. *If G is connected, then the natural morphism $i : \mathtt{Aut}_G(g) \to G$ is an equivalence of groupoids.*

Proof. For every $h \in G$ we choose a morphism $u_h : g \to h$ in G such that $u_g = \mathtt{id}_g$. These choices provide a functor $j : G \to \mathtt{Aut}_G(g)$ which sends every object of G to the unique object g of $\mathtt{Aut}_G(g)$, and every morphism $\phi : h \to h'$ between objects of G to $u_{h'} \circ \phi \circ u_h^{-1} \in \mathtt{Aut}_G(g)$. Then we have the equality $j \circ i = \mathtt{id}_{\mathtt{Aut}_G(g)}$. Furthermore, an isomorphism $\mathtt{id}_G \to i \circ j$ is given by $h \mapsto (u_h^{-1} : h \to g)$. $\square$

For a group G we let $G\mathbf{Set}$ denote the category of sets with a (left)-action of G and G-equivariant maps. The objects of $G\mathbf{Set}$ will be called G-sets. Viewing G as a groupoid with one object we have canonical isomorphism of categories

$$G\mathbf{Set} \simeq \mathbf{Fun}(G, \mathbf{Set}) \ .$$

Problem 1.50. *Show this assertion.*

We shall now use the following fact. If $a : H \to G$ is an equivalence of groupoids, then the restriction map

$$a^* : \mathbf{Fun}(G, \mathbf{Set}) \to \mathbf{Fun}(H, \mathbf{Set})$$

is an equivalence of categories, too.

Problem 1.51. *Show this assertion.*

Assume that $a : H \to G$ is a morphism between groupoids. If $C \in \mathbf{Fun}(H, \mathbf{Set})$ is a representation of H, then we shall define a functor between the transport categories

$$\tilde{a} : T(a^*C) \to T(C) \ .$$

An object of $T(a^*C)$ is given by an element $x \in C(a(h))$ for some $h \in H$. The functor $\tilde{a}$ maps this object to the same element x which is now considered as an object of $T(C)$ over $a(h)$. Let now $x_i \in C(a(h_i))$ for $i = 0, 1$ be two objects of $T(a^*C)$ and $\phi : x_0 \to x_1$ be a morphism in $T(a^*C)$. Then by definition $\phi : h_0 \to h_1$ is a morphism in H is such that $C(a(\phi))(x_0) = x_1$. The functor $\tilde{a}$ maps ϕ to $a(\phi)$.

Problem 1.52. *Show that $\tilde{a}$ is a functor. Further show that for a second morphism $b : K \to H$ between groupoids we have an isomorphism $\tilde{a} \circ \tilde{b} \cong \widetilde{a \circ b} : T((a \circ b)^*C) \to T(C)$.*

Let $a, b : H \to G$ be two morphisms and $\rho : a \to b$ be a natural transformation. Note that ρ associates to every $h \in H$ a morphism $\rho(h) : a(h) \to b(h)$ in G. Then we get a natural isomorphism $\tilde{\rho} : T(a^*C) \to T(b^*C)$ as follows. This transformation maps the object $x \in C(a(h))$ of $T(a^*C)$ to the object $C(\rho(h))(x) \in C(b(h))$ of $T(b^*C)$. On morphisms $\tilde{\rho}$ is given by the identity.

Problem 1.53. *Check details.*

Problem 1.54. *Conclude that an equivalence of groupoids $a : H \to G$ induces an equivalence of transport categories $\tilde{a} : T(a^*C) \to T(C)$.*

From now on we consider a path-connected and locally simply-connected space Y with basepoint y. In order to save notation we write $\pi := \pi_1(Y, y)$.

Corollary 1.55. *If Y is a path-connected and locally simply-connected space, then we have an equivalence*

$$\Phi_y : \mathbf{Cov}(Y) \stackrel{\simeq}{\to} \pi\mathbf{Set} \ .$$

Proof. We compose the equivalences

$$\mathbf{Cov}(Y) \stackrel{\simeq}{\to} \mathbf{Fun}(\Pi(Y), \mathbf{Set}) \stackrel{\simeq}{\to} \mathbf{Fun}(\pi, \mathbf{Set}) \cong \pi\mathbf{Set}$$

given by Φ, restriction along $\pi \to \Pi(Y)$ and the canonical identification. □

We explain the functor

$$\Phi_y : \mathbf{Cov}(Y) \to \pi\mathbf{Set}$$

explicitly. It sends the covering $f : X \to Y$ to the set $\Phi_y(f) := f^{-1}(y)$. The action of $[\gamma] \in \pi_1(Y, y)$ on this set sends $x \in f^{-1}(y)$ to the endpoint of the lift of γ with beginning in x. On morphism this functor is defined in the obvious way.

1.7 Properties of coverings

In this subsection we translate properties of coverings to algebraic properties of the corresponding G-sets. Our standing hypothesis is that Y is a path-connected, locally-simply connected space and $y \in Y$ is a base-point. We abbreviate $\pi := \pi_1(Y, y)$.

Let $S \in \pi\mathbf{Set}$ and $f : X \to Y$ be the corresponding covering. By Problem 1.54 the path groupoid of X is equivalent the transport category of the representation $\pi \to \mathbf{Set}$ corresponding to the π-set S. The objects of the latter transport category are just the elements of S. The morphisms $s \to s'$ are the elements $g \in \pi$ such that $gs = s'$. Note that we can identify $f^{-1}(y) \cong S$. We write $\pi_s \subseteq \pi$ for the stabilizer subgroup of s. We immediately conclude:

Corollary 1.56. *Assume that $f : X \to Y$ is a covering associated to the π-set S.*

1. *We have a natural bijection $\pi\backslash S \cong \pi_0(X)$ which sends the orbit of $s \in S$ to the path-component of the point $s \in X$.*
2. *We have an isomorphism $\pi_1(X, s) \cong \pi_s$.*

Proof. Indeed, $\pi_0(X)$ is in bijection with the connected components of the translation groupoid of the representation S of π. This is just the set of π-orbits.

The fundamental group of X at the base point s is the automorphism of the point s considered as an object of the fundamental groupoid of X, hence of the translation groupoid of the representation S of π. The latter is exactly the stabilizer of the point s. $\square$

Note that π with the left-multiplication is a free and transitive π-set. Up to isomorphism there is a unique transitive and free π-set. Such a π-set is also called a (left) **π-torsor**. Indeed, if S is a π-torsor, then we fix a point $s \in S$ and obtain an isomorphism of π-sets $\pi \cong S$ by $\pi \ni g \mapsto gs \in S$. The group of automorphisms of the π-set π is isomorphic to π acting via right-multiplication.

Corollary 1.57. *Up to isomorphism there is a unique connected and simply-connected covering of Y. Its group of automorphisms is isomorphic to π.*

We will call such a covering a **universal covering** of Y. The action of the automorphisms of the universal covering will be conidered as a right-action of π.

We choose a universal covering $\tilde{Y} \to Y$. We can then construct a functor

$$\Psi_{\tilde{Y}} : \pi\mathbf{Set} \to \mathbf{Cov}(X)$$

which associates to a π-set S the covering

$$\tilde{Y} \times_\pi S \to Y \ .$$

On morphisms the functor is defined in the obvious way.

Lemma 1.58. *The functor $\Psi_{\tilde{Y}} : \pi\mathbf{Set} \to \mathbf{Cov}(Y)$ realizes an inverse equivalence to Φ_y.*

Proof. Let S be a π-set. We consider the π-set $\pi \times S$ on which π acts only on the left factor by left-multiplication. The group π then also acts on this π-set by automorphisms via the right-action on π and the left-action on S. There is a natural isomorphism of π-sets $\pi \times_\pi S \to S$ given by $[g, s] \mapsto gs$.

By Example 1.44 the covering associated to the π-set $\pi \times S$ is the covering $\tilde{Y} \times S \to Y$ given by the projection to the first factor composed with the map $\tilde{Y} \to Y$. The group π then acts on this covering by automorphisms via the right-action on $\tilde{Y}$ and the left action on S. Since an equivalence of categories preserves quotients by actions by automorphisms it is clear that the covering $\tilde{Y} \times_\pi S \to Y$ corresponds to the π-set $\pi \times_\pi S$. $\square$

Let G be a group acting freely and properly from the right on a simply-connected and locally simply-connected space X. We set $Y := X/G$. Then the natural projection $X \to Y$ is a covering. Note that Y is connected, since it the image of a connected space. Furthermore, since X is locally simply-connected and $X \to Y$ is a local homeomorphism, Y is locally simply-connected, too.

Let $y \in Y$.

Corollary 1.59. *The choice of a base-point $x \in X$ over y determines an isomorphism $\pi_1(Y, y) \cong G$.*

Proof. Let $f : X \to Y$ be the projection and note that this is a universal covering of Y. The fibre $f^{-1}(y)$ is a torsor over the groups G and $\pi_1(Y, y)$. The choice of a base-point $x \in f^{-1}(y)$ provides a trivialization of torsors and hence a bijection $G \cong f^{-1}(y) \cong \pi_1(Y, y)$ which maps $g \in G$ to $[\gamma] \in \pi_1(Y, y)$ such that $xg = x[\gamma]$. One can check that this is a homomorphism.

Problem 1.60. *Fill in the details.*

□

Corollary 1.59 allows to calculate the fundamental groups of some spaces.

Example 1.61. Note that $\mathbb{R}^n$ is simply-connected and locally simply-connected. The group $\mathbb{Z}^n$ acts freely and properly by translations and the quotient $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus. Hence $\pi_1(T^n, 0) \cong \mathbb{Z}^n$.

Example 1.62. We will see later (Example 1.71), that the n-dimensional sphere S^n is simply connected for $n \geq 2$. The group $\mathbb{Z}/2\mathbb{Z}$ acts freely on S^n by the antipodal reflection. The quotient $\mathbb{RP}^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$ is the real projective space. Hence $\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.63. We fix two two integers p, q such that $(p, q) = 1$. The group $\mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{C}^2$ by

$$[n](z_1, z_2) \mapsto (e^{2\pi i \frac{n}{p}} z_1, e^{2\pi i \frac{nq}{p}} z_2) \ .$$

This restricts to a free action on the unit sphere $S^3 \subseteq \mathbb{C}^2$. The quotient $L(p, q) \cong S^3/(\mathbb{Z}/p\mathbb{Z})$ is called a lens space. Hence $\pi_1(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$.

Example 1.64. Let G be a semi-simple Lie group of non-compact type, e.g. $SO(p, q)$ for $p, q \geq 1$ and $K \subset G$ be a maximal compact subgroup. The quotient $X := G/K$ is called the symmetric space of G. It is a left G-manifold. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of g. Then it is known that the exponential map of G induces a diffeomorphism

$$\mathfrak{p} \cong X \ , \quad p \mapsto \exp(p)K \ .$$

If $\Gamma \subset G$ is a torsion-free discrete subgroup, then Γ acts properly and freely on X. The quotient $Y := \Gamma\backslash X$ is a locally symmetric space. We have $\pi_1(Y, y) \cong \Gamma$.

Problem 1.65. *Let M be a connected smooth manifold. Show that M admits a universal covering $\tilde{M} \to M$ and that $\tilde{M}$ has a unique smooth manifold structure such that $\tilde{M} \to M$ is smooth.*

Assume that $X \to Y$ is a covering on which a group G acts by automorphisms from the right such that $X/G \cong Y$. Such a covering is called a **Galois covering** for the group G.

Example 1.66. The universal covering of Y is a Galois covering for the group π.

Under the equivalence $\mathbf{Cov}(Y) \simeq \pi\mathbf{Set}$ Galois coverings for G correspond to right G-torsors S with a left action of π. An obvious example is given by a homomorphism $\pi \to G$ where π acts on G via left-multiplication and the homomorphism, and G acts on the right via right-multiplication.

Problem 1.67. *Show that every G-torsor with an action π is isomorphic to one of this form.*

Connected Galois coverings correspond to right G-torsors on which π acts transitively. Note that transitive π-sets are of the form π/H for the subgroup $H \subseteq \pi$. Let us calculate the group of automorphisms of this π-set.

Let G be a group and $H \subseteq G$ be a subgroup. Let $N_G(H)$ denote the normalizer of H in G.

Lemma 1.68. *We have an isomorphism $\mathtt{Aut}_{G\mathbf{Set}}(G/H) \cong H \backslash N_G(H)$.*

Proof. An automorphism ϕ of the G-set G/H is fixed by the image $\phi(H)$ of the class H. So there exists $g \in G$ such that $\phi(H) = gH$. Since ϕ is a map of G-sets we have $gH = \phi(H) = \phi(kH) = k\phi(H)) = kgH$ for all $k \in H$. We get $gH = HgH$ and hence $g \in N_G(H)$. Note that the class $gH = Hg$ is uniquely determined by ϕ. The action of ϕ on G/H is best viewed as given by right-multiplication by the class Hg. □

It is now clear that $H \backslash N_G(H)$ acts transitively on G/H if and only if $N_G(H) = G$, i.e. H is normal.

Corollary 1.69. *Assume that $Z \to Y$ is a connected Galois covering. Then it is of the form $Z \cong \tilde{Y}/H$ for a normal subgroup $H \subset \pi$ and a universal covering $\tilde{Y} \to Y$. In this case $H \cong \pi_1(Z, z)$ and the Galois group of the covering $Z \to Y$ is the quotient $H \backslash \pi$.*

Proof. The connected Galois covering corresponds to a π-set π/H with $H \cong \pi_1(Z, z)$ and where H is normal. Then $\mathtt{Aut}_{\pi\mathbf{Set}}(\pi/H) \cong H \backslash \pi$ is the Galois group. □

1.8 The analogy with Galois theory

In order to avoid the discussion of topological Galois groups we consider a field k whose separable closure $\bar{k}$ is a finite extension of k.

Covering theory	Galois theory
connected and locally simply connected space Y	field k
connected covering $X \to Y$	separable field extension $k \to K$
universal covering $\tilde{Y} \to Y$	separable closure $k \to \bar{k}$
fundamental group $\pi_1(Y, y)$	Galois group $Gal(\bar{k}\vert k)$
non-connected covering $$\bigsqcup_j X_j \to Y$$ for a collection of connected coverings $X_j \to X$	ring extension of the form $$k \to \prod_j K_j$$ for a collection of field extensions $k \to K_j$
correspondence between subgroups $H \subseteq \pi_1(Y, y)$ and intermediate coverings $$\tilde{Y} \to X := \tilde{Y}/H \to Y$$ with $\pi_1(X) \cong H$	correspondence between subgroups $H \subseteq Gal(\bar{k}\vert k)$ and intermediate fields $$k \to K = \bar{k}^H \to \bar{k}$$ with $Gal(\bar{k}\vert K) \cong H$
correspondence between normal subgroups $H \subseteq \pi_1(Y, y)$ and Galois coverings $$X := \tilde{Y}/H \to Y$$ with $$\mathtt{Aut}_{\mathbf{Cov}(Y)}(X \to Y) \cong \pi_1(Y, y)/H$$	correspondence between normal subgroups $H \subseteq Gal(\bar{k}\vert k)$ and field extensions $$k \to K = \bar{k}^H$$ with $$Gal(K\vert k) \cong H$$

1.9 Van Kampen

The van Kampen theorem calculates the fundamental group of a pointed space in terms of fundamental groups of the pieces of a decomposition into open subsets. It uses the notion of a **free product** of a collection $(G_\alpha)_{\alpha\in I}$ of groups . The free product

$$G := *_{\alpha\in I} G_\alpha$$

is the coproduct of this collection of groups in the category **Groups** of groups. Therefore it is a group together with homomorphisms $G_\alpha \to G$ for all $\alpha \in I$ such that the restrictions along these homomorphisms induce a bijection

$$\mathtt{Hom}_{\mathbf{Groups}}(G, H) \cong \prod_{\alpha\in I} \mathtt{Hom}_{\mathbf{Groups}}(G_\alpha, H)$$

for every group H. The free product has the following explicit description in terms of generators and relations. An element in G can be represented as a word $g_r \dots g_1$ for some $r \in \mathbb{N}$, where $g_i \in G_{\alpha_i}$ for a collection $\alpha_1, \dots, \alpha_r \in I$. If $\alpha_i = \alpha_{i+1}$, then $g_r \dots (g_{i+1} g_i) \dots g_1$ represents the same element of G. Moreover, if $g_i = 1$, then $g_r \dots g_{i+1} g_{i-1} \dots g_1 = g_r \dots g_1$. The group G can be considered as generated by the words as above subject to this type of relations.

Let X be a topological space and $x \in X$ be a base point. We consider a decomposition $X = \bigcup_{\alpha\in I} U_\alpha$ into open subsets such that the multiple intersections U_α, $U_\alpha \cap U_\beta$, and $U_\alpha \cap U_\beta \cap U_\gamma$ are path-connected for all $\alpha \in I$, $(\alpha, \beta) \in I^2$, $(\alpha, \beta, \gamma) \in I^3$, respectively. For the base point we assume $x \in \bigcap_{\alpha\in I} U_\alpha$.

Theorem 1.70 (van Kampen)**.** *The fundamental group $\pi_1(X, x)$ is given as the quotient of the free product $*_{\alpha\in I}\pi_1(U_\alpha, x)$ by the normal subgroup N generated by the elements $f_\alpha^{-1}(g) f_\beta(g)$ for all pairs $(\alpha, \beta) \in I^2$ and $g \in \pi_1(U_\alpha \cap U_\beta, x)$, where $f_\alpha : \pi_1(U_\alpha \cap U_\beta, x) \to \pi_1(U_\alpha, x)$ denotes the map induced by the inclusion $U_\alpha \cap U_\beta \hookrightarrow U_\alpha$.*

Proof. By the universal property of the free product the collection of homomorphisms $\pi_1(U_\alpha, x) \to \pi_1(X, x)$ for all $\alpha \in I$ induces a homomorphism

$$\tilde\phi : *_{\alpha\in I}\pi_1(U_\alpha, x) \to \pi_1(X, x) \ .$$

It is clear that

$$\tilde\phi(f_\alpha^{-1}(g) f_\beta(g)) = \tilde\phi(f_\alpha^{-1}(g))^{-1} \tilde\phi(f_\beta(g)) = 1$$

since $g \in \pi_1(U_\alpha \cap U_\beta, x)$ is mapped to the same element under the two maps

$$U_\alpha \cap U_\beta \to U_\alpha \to X \ , \quad U_\alpha \cap U_\beta \to U_\beta \to X \ .$$

Hence $\tilde\phi$ factorizes through the quotient by N:

$$\begin{tikzcd} *_{\alpha\in I}\pi_1(U_\alpha, x) \arrow[r, "\tilde\phi"] \arrow[d] & \pi_1(X, x) \\ *_{\alpha\in I}\pi_1(U_\alpha, x)/N \arrow[ur, dotted, "\phi"] & \end{tikzcd} \ .$$

We show that ϕ is surjective. Let $[\gamma] \in \pi_1(X, x)$. Then we can write the path γ as a finite concatenation $\gamma_r \circ \dots \circ \gamma_1$ of paths such that γ_i is a path in U_{α_i}. We choose paths σ_i from x to the endpoint of γ_i for $i = 1, \dots, r-1$. We let σ_0 and σ_r be the constant paths at x. We get loops $\kappa_i := \sigma_i^{-1} \circ \gamma_i \circ \sigma_{i-1}$ in U_{α_i}. Then γ is homotopic to $\kappa_r \circ \dots \circ \kappa_1$. Here we use that the loop $\sigma_i \circ \sigma_i^{-1}$ is homotopic to the constant path at the endpoint of γ_i. Hence $[\gamma] = \tilde{\phi}([\kappa_r]) \circ \dots \circ \tilde{\phi}([\kappa_1])$.

It remains to show that ϕ is injective.

An general element g in $*_{\alpha \in I} \pi_1(U_\alpha, x)/N$ is represented by a word $g_r \circ \dots \circ g_1$ with $g_i \in \pi_1(U_{\alpha_i}, x)$ for a given sequence $\alpha_1, \dots, \alpha_r$. We call this a factorization of g. We call another factorization of g related, if one of the following holds:

1. There is an index i such that $\alpha_i = \alpha_{i+1}$ and the other factorization is given by
$$g_r \circ \dots \circ (g_{i+1} g_i) \circ \dots \circ g_1 \ ,$$
2. There is an index i and $h \in \pi_1(U_{\alpha_i} \cap U_\beta, x)$ such that $g_i = f_{\alpha_i}(h)$ and the other factorization is obtained by replacing g_i by $f_\beta(h)$ and α_i by β.

We consider the equivalence relation on factorizations generated by the relation *related*. One easily checks that the first sort of relation accounts for the relation of factorizations in the free product, and the second sort accounts for the quotient by N. In other words, $*_{\alpha \in I} \pi_1(U_\alpha, x)/N$ is in bijection with the set of equivalence classes of factorizations.

We now consider two factorizations $[\gamma] = [\gamma_r] \circ \dots [\gamma_1]$ and $[\gamma'] = [\gamma'_{r'}] \circ \dots [\gamma'_1]$ such that $\phi([\gamma]) = \phi([\gamma'])$. Then there exists a homotopy $h : I \times I \to X$ such that $h(0, t) = \gamma(t)$, $h(1, t) = \gamma'$. We can choose a partition $0 = s_0 < s_1 \dots < s_m = 1$ of I, for every $j = 0, \dots, m-1$ a partition $0 = t_0^j < t_1^j \dots < t_{r_j}^j = 1$, and $\alpha_{ji} \in I$ such that the rectangle R_{ji} with corners (s_j, t_i^j), (s_j, t_{i+1}^j), (s_{j+1}, t_{i+1}^j), (s_{j+1}, t_i^j) is mapped to $U_{\alpha_{ji}}$, and such that the induced factorization of $h_{|\{0\} \times I}$ is the factorization of γ, and the factorization of $h_{|\{1\} \times I}$ is the factorization of γ'. We can further arrange by shifting the t_i^j slightly that every corner belongs to at most three rectangles. For every corner (s_j, t_i^j) we choose a path from $h(s_j, t_i^j)$ to x inside the intersection of the at most three of sets $U_{\alpha_{j'i'}}$ which contain the rectangles containing the corner. We use these paths to extend the restriction of h to boundary faces of the rectangles to loops.

We count the rectangles lexicographically. We let u_k be the factorized path in I^2 from $(0, 0)$ to $(1, 1)$ which separates the first k rectangles $R_1, \dots, R_k$ from the remaining. Then $h_{|u_0}$ gives (after completing the interior pieces to loops) the factorization of γ, and $h_{|u_k}$ for sufficiently large k is the factorization of γ'. The path u_k presents a factorization of $[\gamma]$ which depends on the choice of a suitable $\alpha \in I$ for every segment.

The transition from u_k to u_{k+1} is given by sliding over R_{k+1} On can check that this transition replaces the factorizations by an equivalent one. First one replaces the choices open subsets U_α for the boundary segments of the rectangle by an $\beta \in I$ such that

$h(R_{k+1}) \subset U_\beta$. Using the second type of generators of the relation twice (for the horizontal and the vertical boundary component) we check that we stay in the equivalence class of factorization. Then we use the first type of generators relation in order to check that sliding over fixes the equivalence class. $\square$

Example 1.71. We assume that $n \geq 2$. We cover S^n by the complements U and V of the north- and south pole. We pick a base point in the equator S^{n-1}. Then $U \cap V$ is homotopy equivalent to S^{n-1}, and U and V are contractible. Since S^{n-1} is path-connected we can apply van Kampen's Theorem. We conclude that $\pi_1(S^n) \cong \{1\} *_{\pi_1(S^{n-1})} \{1\} \cong \{*\}$. In other words, S^n is simply connected for $n \geq 2$.

Example 1.72. For $g, \ell \in \mathbb{N}$ we let $\Sigma_{g,\ell}$ be an oriented surface of genus g and ℓ boundary circles. In particular, $\Sigma_{1,1}$ is the complement of a closed disc in the two-torus T^2, i.e. $\Sigma_{1,1} \cong T^2 \setminus D^2$. If we represent T^2 as a square with sides identified appropriately and the disc in the interior of the square, then we easily see that $T^2 \setminus D^2$ is homotopy equivalent to the space obtained by removing the whole interior of the square. This is a wedge of two circles $S^1 \vee_1 S^1$. They contribute generators a, b to the fundamental group. By van Kampen $\pi_1(S^1 \vee_1 S^1, 1) \cong \pi_1(\Sigma_{1,1}, *)$ is the free group in these two generators. The boundary circle of $\Sigma_{g,\ell}$ is the commutator $[a, b] := aba^{-1}b^{-1}$.

The **free group** in n generators will be denoted by $\mathbb{F}_n$. Furthermore we use the notation

$$\langle a_1, \ldots, a_s \mid R_1, \ldots, R_r \rangle$$

for the group generated by $a_1, \ldots, a_s$ with the relations $R_1, \ldots, R_r$, where R_i are words in the generators.

Example 1.73. For example

$$\mathbb{F}_2 \cong \langle a, b, c \mid [a, b]c \rangle \ , \quad \mathbb{Z}^2 \cong \langle a, b \mid [a, b] \rangle$$

Example 1.74. We now calculate the fundamental groups of $\Sigma_{g,\ell}$ for all $g, \ell \in \mathbb{N}$. We start with $\Sigma_{0,\ell}$. This is a closed disc with $\ell - 1$ small open discs removed. It is obviously homotopy equivalent to a wedge of the $\ell - 1$ boundary circles of the interior discs. We let c_i for $i = 2, \ldots, \ell$ be the interior boundary circles run counter-clockwise, and c_1 be the exterior circle run clockwise. It follows that $\pi_1(\Sigma_{0,\ell})$ is generated by $c_1, \ldots, c_\ell$ with the relation $c_\ell \ldots c_1 = 1$. We claim that by induction that

$$\pi_1(\Sigma_{g,\ell}) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_1 \ldots, c_\ell \mid [a_1, b_1] \ldots [a_g, b_g] c_\ell \ldots c_1 \rangle \ .$$

As long as $\ell \geq 1$, this group is free in a_i, b_i and $c_2, \ldots, c_{\ell-1}$. Indeed if we glue in $\Sigma_{1,1}$ to $\Sigma_{g,\ell}$ along the ℓ'th boundary circle, then we get $\Sigma_{g+1,\ell-1}$. In this process we add generators a_{g+1}, b_{g+1} and the relation $c_\ell = [a_{g+1}, b_{g+1}]$.

If $\ell = 0$, then

$$\pi_1(\Sigma_{g,0}) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] \rangle \ .$$

This group is not free.

The fundamental groupoid and the fundamental groups behave well with respect to products. The isomorphisms below are induced by the the projection to the functors.

Problem 1.75. *We have an isomorphism* $\Pi(X \times Y) \cong \Pi(X) \times \Pi(Y)$*.*

Problem 1.76. *We have an isomorphism* $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$*.*

Example 1.77. For $\ell, \ell' \in \mathbb{N} \setminus \{0\}$ we have

$$\pi_1(\Sigma_{0,\ell} \times \Sigma_{0,\ell'}) \cong \mathbb{F}_\ell \times \mathbb{F}_{\ell'} \ .$$

This group is not free.

1.10 Flat vector bundles

We first recall some facts from the theory of vector bundles and connections. We consider a smooth manifold M and a real vector bundle $V \to M$. Let us fix a connection ∇ on V. If γ is a smooth path, then we have a parallel transport along γ, a linear isomorphism

$$P_\gamma : V_{\gamma(0)} \to V_{\gamma(1)}$$

from the fibre of V over $\gamma(0)$ to the fibre of V over $\gamma(1)$. Assume that μ is a second smooth path starting in $\gamma(1)$ which is such that the concatenation $\mu \circ \gamma$ is smooth. Then we have the identity

$$P_{\mu\circ\gamma} = P_\mu \circ P_\gamma \ .$$

If the connection is flat (i.e. its curvature $R^\nabla := \nabla^2$ vanishes), then the parallel transport P_γ only depends on the smooth homotopy class of γ.

We let $\Pi^\infty(M)$ be the version of the fundamental groupoid based on smooth paths which are constant near the endpoints in order to have a smooth concatenation and smooth homotopies. Then there is a natural morphism

$$\Pi^\infty(M) \to \Pi(M) \ .$$

Fact 1.78. *This morphism is an isomorphism.*

Proof. To this end we must show that every continuous path is homotopic to a smooth path which is constant near the endpoints, and that every continuous homotopy between smooth paths can be replaced by a smooth homotopy. The arguments belong to the field of differential topology and will not be given here. □

In the following we will always assume that γ is a smooth representative of the morphism $[\gamma] \in \Pi(M)$

A vector bundle with a flat connection will be called a **flat vector bundle** and denoted by (V, ∇). A morphism of flat vector bundles $(V, \nabla^V) \to (W, \nabla^W)$ is a morphism of vector bundles $\phi : V \to W$ which preserves the connections. Note that the two connections ∇^V and ∇^W induce a connection $\nabla^{\mathtt{Hom}(V,W)}$ on the bundle $\mathtt{Hom}(V, W) \to M$. The condition that ϕ preserves the connections is equivalent to $\nabla^{\mathtt{Hom}(V,W)}\phi = 0$. We form the category $\mathbf{Bun}_{\mathbb{R}}^{flat}(M)$ of flat vector bundles on M and connection preserving bundle morphisms.

Let $\mathbf{Vect}_{\mathbb{R}}^{fin}$ be the category of finite-dimensional real vector spaces.

Corollary 1.79. *A flat vector bundle induces a representation*

$$\Phi(V, \nabla^V) : \Pi(M) \to \mathbf{Vect}_{\mathbb{R}}^{fin} \ , \quad m \mapsto V_m \ , \quad [\gamma] \mapsto P_\gamma \ .$$

This construction provides a functor

$$\Phi : \mathbf{Bun}_{\mathbb{R}}^{flat}(M) \to \mathbf{Fun}(\Pi(M), \mathbf{Vect}_{\mathbb{R}}^{fin}) \ .$$

Problem 1.80. *Verify the details.*

Problem 1.81. *If $N \to M$ is a smooth map, then we have a functor*

$$f^* : \mathbf{Bun}_{\mathbb{R}}^{flat}(M) \to \mathbf{Bun}_{\mathbb{R}}^{flat}(N) \ .$$

Show that there is a natural equivalence between the two compositions in

$$\begin{tikzcd} \mathbf{Bun}_{\mathbb{R}}^{flat}(M) \arrow[r, "f^*"] \arrow[d, "\Phi_M"] & \mathbf{Bun}_{\mathbb{R}}^{flat}(N) \arrow[d, "\Phi_N"] \\ \mathbf{Fun}(\Pi(M), \mathbf{Vect}_{\mathbb{R}}^{fin}) \arrow[r, "\Pi(f)^*"] & \mathbf{Fun}(\Pi(N), \mathbf{Vect}_{\mathbb{R}}^{fin}) \end{tikzcd} \ .$$

Lemma 1.82. *The functor*

$$\Phi : \mathbf{Bun}^{flat}_{\mathbb{R}}(M) \to \mathbf{Fun}(\Pi(M), \mathbf{Vect}^{fin}_{\mathbb{R}}) \ .$$

is an equivalence of categories.

Proof. We construct the inverse equivalence Ψ. It is similar to the construction of Ψ in Subsection 1.5. Let $C : \Pi(M) \to \mathbf{Vect}^{fin}_{\mathbb{R}}$ be given. Then the underlying set of the total space of the bundle $\Psi(C) \to M$ is $\bigsqcup_{m\in M} C(m)$. This set has a projection to M whose fibres are real vector spaces $C(m)$, $m \in M$. In order to define the manifold structure we use the local trivializations as in Subsection 1.5. They are compatible with the $\mathbb{R}$-vector space structures on the fibres. The transition functions $(y, a) \mapsto \phi(y, a)$ are locally constant, and $(a \mapsto \phi(y, a)) \in \mathtt{Aut}_{\mathbf{Vect}^{fin}_{\mathbb{R}}}(C(y))$. Hence we get a real vector bundle. It has a unique connection which becomes the trivial connection in the trivializations used in the constructions.

Problem 1.83. *Complete the construction of* Ψ *and show that it can be used as an inverse equivalence to* Φ.

Let G be a group. Then $\mathbf{Rep}^{fin}_{\mathbb{R}}(G)$ denotes the category of representations of G on finite-dimensional real vector spaces. We have an isomorphism of categories

$$\mathbf{Rep}^{fin}_{\mathbb{R}}(G) \cong \mathbf{Fun}(G, \mathbf{Vect}^{fin}_{\mathbb{R}}) \ .$$

We now assume that M is connected and that $m \in M$ is a base point. Then we get the chain of equivalences

$$\mathbf{Bun}^{flat}_{\mathbb{R}}(M) \simeq \mathbf{Fun}(\Pi(M), \mathbf{Vect}^{fin}_{\mathbb{R}}) \simeq \mathbf{Fun}(\pi_1(M, m), \mathbf{Vect}^{fin}_{\mathbb{R}}) \cong \mathbf{Rep}^{fin}_{\mathbb{R}}(\pi_1(M, m)) \ .$$

Corollary 1.84. *If* M *is a connected manifold and* $m \in M$ *is a base point, then we have an equivalence*

$$\Phi_y : \mathbf{Bun}^{flat}_{\mathbb{R}}(M) \stackrel{\sim}{\to} \mathbf{Rep}^{fin}_{\mathbb{R}}(\pi_1(M, m))$$

which associates to a flat vector bundle (V, ∇) *the representation of* $\pi_1(M, m)$ *on the fibre* V_m *induced by the parallel transport called the* ***holonomy representation***.

We now describe the inverse equivalence $\Psi_{\tilde{M}}$ which depends on the choice of a universal covering $\tilde{M} \to M$. Note that the universal covering is a manifold by Problem 1.65. Let (ρ, U) be a representation of $\pi := \pi_1(M, m)$ on a finite-dimensional vector space U. Then we define the vector bundle $V := \tilde{M} \times_\pi U \to M$. The trivial connection on the bundle $\tilde{M} \times U \to \tilde{M}$ is π-invariant and descends to the flat connection ∇^V on V. We set $\Psi_{\tilde{M}}(\rho, U) := (V, \nabla^V)$.

Problem 1.85. *Fill in the missing details to show that $\Psi_{\tilde{M}}$ is an inverse equivalence to Φ_y.*

Example 1.86. We get a classification of flat vector bundles on S^1. Note that $\pi_1(S^1, 1) \cong \mathbb{Z}$. A representation ρ on a vector space U is uniquely determined by $\rho(1)$. A representation ρ' on the same vector space is isomorphic to ρ, if there exists $g \in \mathtt{Aut}_{\mathtt{Vect}_{\mathbb{R}}^{fin}}(U)$ such that $g\rho(1)g^{-1} = \rho'(1)$.

For a group G we let $C(G)$ denote the set of conjugacy classes in G. We conclude that there is a bijection

$$\mathtt{Bun}_{\mathbb{R}}^{flat}(S^1)/iso \cong \bigsqcup_{n\in\mathbb{N}} C(GL(n, \mathbb{R})) \ .$$

For example, one-dimensional flat bundles are classified by the set $\mathbb{R}^* \cong C(Gl(1, \mathbb{R}))$.

Example 1.87. Note that for $\ell \geq 1$ we have

$$\pi_1(\Sigma_{g,\ell}) \cong \mathbb{F}_{2g+\ell-1} \ .$$

Therefore flat n-dimensional vector bundles on a surface $\Sigma_{g,\ell}$ are classified by the set

$$Gl(n, \mathbb{R})^{2g+\ell-1}/Gl(n, \mathbb{R}) \ ,$$

where $Gl(n, \mathbb{R})$ acts diagonally by conjugation.

The set of flat n-dimensional vector bundles on $\Sigma_{g,0}$ is classified by

$$\{(a_1, \ldots, a_g, b_1, \ldots, b_g) \in Gl(n, \mathbb{R})^{2g} \mid [a_1, b_1] \ldots [a_g, b_g] = 1\}/Gl(n, \mathbb{R}) \ ,$$

where $Gl(n, \mathbb{R})$ again acts by conjugation on all factors.

For $g = 1$ this is the set of conjugacy class of pairs of commuting elements in $Gl(n, \mathbb{R})$.

These sets have additional structures (topological spaces, singular varieties) and are very interesting mathematical objects.

Problem 1.88. *Let $GL(n, \mathbb{R})^\delta$ be the group $GL(n, \mathbb{R})$ equipped with the discrete topology. Show that flat n-dimensional real vector bundles on M correspond to Galois coverings of M for the group $GL(n, \mathbb{R})^\delta$.*

1.11 Lifting of maps

In this subsection we consider the following problem. Assume that we are given bold part of the diagram

$$\begin{tikzcd} \{z\} \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & Y \end{tikzcd}$$

where X, Y, Z are path-connected and f is a covering. We are interested whether a lift $\tilde{g}$ exists.

Let $x \in X$ be the image of the point z in X, and $y := f(x) = g(z)$. If a lift exists, then we have

$$\pi_1(g) = \pi_1(f) \circ \pi_1(\tilde{g}) : \pi_1(Z, z) \to \pi_1(Y, y) \ .$$

Hence a necessary condition for the existence of a lift $\tilde{g}$ is that

$$\pi_1(g)(\pi_1(Z, z)) \subseteq \pi_1(f)(\pi_1(X, x)) \ . \tag{1}$$

Proposition 1.89. *We assume that Z is locally path-connected. Then the lift $\tilde{g}$ exists if and only if the condition* (1) *is satisfied. Furthermore, it is unique.*

Proof. Let $z_1 \in Z$. Since Z is path-connected we can choose a path γ in Z from z to z_1. We are forced to define $x_1 := \tilde{g}(z_1)$ as the endpoint of the unique lift of the path $f \circ \gamma$ with beginning in x. So uniqueness is clear. We must check that $\tilde{g}(z_1)$ is well-defined. We choose a second path γ' and obtain the endpoint x_1'. Then $\sigma := \gamma'^{,-1} \circ \gamma$ is a loop at z and $x_1' = [g \circ \sigma](x_1)$. By assumption $[g \circ \sigma] \in \mathtt{im}(\pi_1(f))$. But then the unique lift of $g \circ \sigma$ with beginning in x is closed, hence $x_1 = x_1'$.

We now must show that the map $\tilde{g}$ is continuous. It suffices to show continuity near each point of Z. Because of uniqueness of the lift we can thus assume that we consider the point z. Since f is a covering we can find a neighborhood $U_y \subseteq Y$ of y such that $f^{-1}(U_y) \cong U_y \times F$ for a discrete set F. We let $f \in F$ be such that $x = (y, f)$ under this homeomorphism. Since Z is locally path-connected we can now find a path-connected neighourhood $U_z \subseteq Z$ of z such that $g(U_z) \subset U_y$. Then the restriction of $\tilde{g}$ to U_z is given by $\tilde{g}(z') = (g(z'), f)$ for all $z' \in U_z$. This map is obviously continuous. $\square$

Example 1.90. If Z is a path-connected and locally simply-connected space, then the lifting problem has a unique solution. For example, for $n \geq 2$ the lifting problem

$$\begin{tikzcd} \{N\} \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ S^n \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & Y \end{tikzcd}$$

has a unique solution for every covering f.

Problem 1.91. *Show that if the data of the lifting problem consists of smooth manifolds and smooth maps, then the lift is smooth as well.*

Example 1.92. We consider a path-connected and locally simply-connected topological group G, e.g. a connected Lie group. Then the total space of its universal covering $\tilde{G} \to G$ is a group. To this end we fix a base point $e \in \tilde{G}$ over $1 \in G$. This will be the identity of the group structure. In order to define the multiplication we consider the diagram

$$\begin{tikzcd}
\{(e,e)\} \arrow[rr, "e"] \arrow[d] & & \tilde{G} \arrow[d, "f"] \\
\tilde{G} \times \tilde{G} \arrow[r] \arrow[urr, dotted, "\tilde{g}"] & G \times G \arrow[r, "mult"] & G
\end{tikzcd} .$$

Since the product $\tilde{G} \times \tilde{G}$ is path-connected and locally simply-connected, the map $\tilde{g}$ exists and is unique.

Problem 1.93. *Show that $\tilde{g}$ defines a group multiplication (verify associativity and the existence of inverses).*

If we apply this construction to the group $SO(n)$ for $n \geq 3$, then we obtain the **spin group** $Spin(n)$. It is known that $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. Consequently, $Spin(n) \to SO(n)$ is a two-sheeted covering. Using Problem 1.91 and Problem 1.65 we see that $Spin(n)$ is a Lie group in a natural way.

2 Die Homotopiekategorie

2.1 Basic constructions

We let **Top** denote the category of topological spaces. Its objects are topological spaces. The morphisms in **Top** are continuous maps.

The category **Top** has a **cartesian product**. The cartesian product of two topological spaces $X, Y \in$ **Top** is usually represented by the space $X \times Y$ whose underlying set is the cartesian product of the underlying sets of X and Y, and the topology is the smallest topology such that the projections to the factors are continuous.

The **coproduct** of two topological spaces X and Y is the disjoint union $X \sqcup Y$ of topological spaces.

Problem 2.1. *Check that these descriptions of products and coproducts are correct.*

The cartesian product and the coproduct are symmetric monoidal structures on **Top**.

We now assume that Y is a locally compact and Hausdorff topological space.

Remark 2.2. By definition, a space Y is locally compact, if every point $y \in Y$ has a compact neighborhood. If Y is in addition Hausdorff, then every point has a basis of compact neighborhoods.

For a second topological space Z we can define a **mapping space**

$$\mathtt{Map}(Y, Z) \in \mathbf{Top} \ .$$

Its underlying set is the set $\mathtt{Hom}_{\mathbf{Top}}(Y, Z)$ of continuous maps from Y to Z. Its topology is the **compact-open topology**. A sub-basis $\{V(K, U)\}_{(K,U)}$ for compact-open topology is indexed by pairs (K, U) of compact subsets $K \subseteq Y$ and open subsets $U \subseteq Z$. The corresponding subset of $\mathtt{Hom}_{\mathbf{Top}}(Y, Z)$ is given by

$$V(K, U) := \{\phi \in \mathtt{Hom}_{\mathbf{Top}}(Y, Z) \mid \phi(K) \subseteq U\} \ .$$

Remark 2.3. We must convince ourselves that the subsets $V(K, U)$ cover $\mathtt{Hom}_{\mathbf{Top}}(Y, Z)$. Thus let $g \in \mathtt{Hom}_{\mathbf{Top}}(Y, Z)$, $y \in Y$ and $U \subseteq Z$ be an open neighborhood of $g(y)$. Then $g^{-1}(U)$ is an open neighborhood of y. Since we assume that Y is locally compact **and** Hausdorff, there exists a compact neighborhood K of y such that $K \subseteq g^{-1}(U)$. Then $g \in V(K, U)$.

Lemma 2.4 (**Exponential law**). *For topological spaces X, Y, Z such that Y is locally compact and Hausdorff we have a natural bijection*

$$\mathtt{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \mathtt{Hom}_{\mathbf{Top}}(X, \mathtt{Map}(Y, Z)) \ .$$

More precisely, we have an adjunction of functors

$$\dots \times Y : \mathbf{Top} \leftrightarrows \mathbf{Top} : \mathtt{Map}(Y, \dots) \tag{2}$$

which is in addition natural in Y.

Proof. Just considering the underlying sets we have a bijection between the sets $Z^{X \times Y}$ and $(Z^Y)^X$ which sends $f : X \times Y \to Z$ to the map

$$g : X \ni x \mapsto (Y \ni y \mapsto f(x, y) \in Z) \ .$$

We must show that f is continuous if and only if g takes values in the set of continuous maps from Y to Z and is continuous as a map from X this mapping space.

Let us first assume that f is continuous. Then $g(x) = f(x, \dots) : Y \to Z$ is continuous for every $x \in X$. We fix a compact $K \subseteq Y$ and an open $U \subseteq Z$. Then we must show that $g^{-1}(V(K,U)) \subseteq X$ is open. Now

$$g^{-1}(V(K,U)) = \{x \in X \mid f(\{x\} \times K) \subseteq U\} \ .$$

Let $x \in g^{-1}(V(K,U))$. We must show that there exists an open neighborhood $Q \subseteq X$ of x such that $Q \subseteq g^{-1}(V(K,U))$. Since f is continuous, for every $k \in K$ we find open neighborhoods $W_k \subseteq Y$ of k and $Q_k \subseteq X$ of x such that $f(Q_k \times W_k) \subseteq U$. Since K is compact, there is a finite subset $I \subseteq K$ such that $K \subseteq \bigcup_{k \in I} W_k$. Then $Q := \bigcap_{k \in I} Q_k$ is an open neighbourhood of x and $f(Q \times K) \subseteq U$. Hence $Q \subseteq g^{-1}(V(K,U))$. We conclude that $g^{-1}(V(K,U))$ is open.

Let us now assume that g takes values in continuous maps and is continuous. We must show that f is continuous. We fix an open $U \subseteq Z$. We fix $(x,y) \in X \times Y$ such that $f(x,y) \in U$. We must find an open neighborhood $Q \times R \subseteq X \times Y$ of (x,y) such that $f(Q \times R) \subseteq U$. We can choose a compact neighborhood $K \subseteq Y$ of y such that $g(x) \in V(K,U)$, see Remark 2.3. Since g is continuous, there exists an open neighourhood $Q \subseteq X$ of x such that $g(Q) \subseteq V(K,U)$, in other words, $f(Q \times K) \subseteq U$. We can now choose an open neighborhood $R \subseteq K$ of y. □

Remark 2.5. In the proof above we have used that Y is locally compact and Hausdorff in an essential way. Because of this restriction we can not say that **Top** is a closed symmetric monoidal category. It is no solution to restrict the considerations to locally compact Hausdorff spaces since e.g. the mapping space $\mathtt{Map}(I,I)$ between two compact spaces is not locally compact. The solution is to consider the so-called **convenient category** of compactly generated Hausdorff topological spaces $\mathbf{Top}^c$.

Problem 2.6. *Assume that X and Y are Hausdorff and locally compact. Show that we have a natural (in X,Y,Z) homeomorphism*

$$\mathtt{Map}(X, \mathtt{Map}(Y,Z)) \cong \mathtt{Map}(X \times Y, Z) \ . \tag{3}$$

This uses the associativity of the cartesian product.

Problem 2.7. *If Y is a locally compact Hausdorff space and Z is a topological space, then the evaluation map*

$$\mathtt{ev} : Y \times \mathtt{Map}(Y,Z) \to Z$$

is continuous. Again deduce this from the exponential law formally.

Problem 2.8. *If X and Y are locally compact Hausdorff spaces and Z is a topological space, then the composition*

$$\mathtt{Map}(Y,Z) \times \mathtt{Map}(X,Y) \to \mathtt{Map}(X,Z)$$

is continuous.

Example 2.9. The **free loop space** LX of a space X is defined by

$$LX := \mathtt{Map}(S^1, X) \ .$$

Every collection of points $t := (t_\alpha)_{\alpha \in I}$ in S^1 gives an evaluation $\mathtt{ev}_t : LX \to X^I$. The group S^1 acts on LX by

$$S^1 \times LX \to X \ , \quad (s, \gamma) \mapsto (t \mapsto \gamma(t-s) \ .$$

In order to see that this is continuous it is useful to write the action using the formal properties of the mapping space. We start with the action of S^1 on itself

$$a : S^1 \times S^1 \to S^1 \ , \quad (s,t) \mapsto t - s \ .$$

It induces the action map via

$$S^1 \times \mathtt{Map}(S^1, X) \xrightarrow{\mathtt{id}_{S^1} \times a^*} S^1 \times \mathtt{Map}(S^1 \times S^1, X) \overset{(3)}{\cong} S^1 \times \mathtt{Map}(S^1, \mathtt{Map}(S^1, X)) \xrightarrow{\mathtt{ev}} \mathtt{Map}(S^1, X)$$

We have an inclusion $i : X \to LX$ which sends a point $x \in X$ to the constant loop at x. Its image is the set of fixed points of the action of S^1.

The free loop space is an important object of study. In string theory for example, it models the configurations of a string in X. The movement of a string in X is a path in LX.

2.2 Pairs and pointed spaces

A **pair** of topological spaces (X, A) is a pair of a space X and a subspace $A \subseteq X$. A morphism between pairs $(X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. We let $\mathbf{Top}^2$ denote the category of pairs of topological spaces.

In particular, if A is just a point $\{x\}$, then we call $(X, \{x\})$ a **pointed space** and write (X, x), or sometimes X, if the base point is understood implicitly. We let $\mathbf{Top}_*$ denote the category of pointed spaces. We have a functor

$$\mathbf{Top}^2 \to \mathbf{Top}_* \ , \quad (X, A) \mapsto X/A \ ,$$

where X/A is defined as the colimit of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ \{*\} & \end{tikzcd} .$$

Its base point is the image of $*$.

In the following we concentrate on the pointed case.

The cartesian product in the pointed case can be represented by

$$(X, x) \times (Y, y) \cong (X \times Y, (x, y)) .$$

The coproduct of pointed spaces (X, x) and (Y, y) is given by the **wedge product**. It is the space

$$(X, x) \vee (Y, y) := \frac{X \sqcup Y}{x \sqcup y} .$$

obtained from the disjoint union of X and Y by identifying the two base points x and y to the new base point.

Problem 2.10. *Check these assertions.*

We consider pointed spaces. We discuss the process of taking **quotients in stages**. For a morphism $f : X \to Y$ we use the notation $Y/f(X)$ for the colimit of the push-out diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \{*\} & \end{tikzcd} .$$

The following is an example. We will encounter similar situations occasionally. Let $i : A \to X$ and $j : B \to X$ be morphisms between pointed spaces. Then we get a map $i \vee j : A \vee B \to X$ and an induced map $p : X \to X/i(A)$.

Lemma 2.11. *We have an isomorphism*

$$\frac{X}{(i \vee j)(A \vee B)} \cong \frac{X/i(A)}{(p \circ j)(B)} .$$

Proof. We consider the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & B \arrow[d] & \\
A \arrow[r] \arrow[d] & A \vee B \arrow[r, "i\vee j"] \arrow[d] & X \arrow[d, "q"] \\
* \arrow[r] & B \arrow[r, "k"] \arrow[d] & Z \arrow[d] \\
& * \arrow[r] & Z/k(B)
\end{tikzcd} .$$

We define Z by the condition that the middle right square is a push-out square. We observe that the middle left square is a push-out square, too. For this it is helpful to consider the composition with the upper left square. The composition of the two middle push-out squares is again a push-out square. Hence we can identify $Z \cong X/i(A)$ and q with the projection map p. With this identification, $k = p \circ j$ and the lower right corner is isomorphic to $(X/i(A))/(p\circ j)(B)$. The composition of the two right push-out squares is again a push-out square. Hence we can identify the lower right corner also with $X/(i\vee j)(A\vee B)$. $\square$

We now assume that Y is locally compact and Hausdorff. We define

$$\mathtt{Map}((Y,y),(Z,z)) \in \mathbf{Top}_*$$

to be the subspace of $\mathtt{Map}(Y,Z)$ of maps which preserve the base point. The base point of the mapping space is the constant map with value z.

We want that for locally compact and Hausdorff spaces Y the mapping space functor $\mathtt{Map}((Y,y),\dots)$ has a left-adjoint similarly as in the unpointed case (2). To this end we observe that if $g \in \mathtt{Hom}_{\mathbf{Top}_*}((X,x),\mathtt{Map}((Y,y),(Z,z))$, then $g(x')(y') = z$ provided $(x',y') \in X\times\{y\}\cup\{x\}\times Y$. We are led to define the **smash product** between pointed spaces by

$$(X,x)\wedge(Y,y) := \frac{X\times Y}{X\times\{y\}\cup\{x\}\times Y} .$$

Lemma 2.12. *If Y is locally compact and Hausdorff, then we have an adjunction*

$$\dots\wedge(Y,y) : \mathbf{Top}_* \leftrightarrows \mathbf{Top}_* : \mathtt{Map}((Y,y),\dots)$$

which is natural in Y.

Proof. Exercise. $\square$

Problem 2.13. *Let (X, A) and (Z, C) be pairs of topological spaces. Assume that Z is Hausdorff and locally compact and that C is a closed subspace. Then we have an isomorphism*

$$(X \times Z)/(A \times Z \cup X \times C) \cong (X/A) \wedge (Z/C)$$

in $\mathbf{Top}_*$. *This can formally be deduced from Lemma 2.12.*

Lemma 2.14. *The smash product induces a symmetric monoidal structure on* $\mathbf{Top}_*$.

Proof. Exercise. □

Remark 2.15. The symmetric monoidal structure $\wedge$ on $\mathbf{Top}_*$ should not be confused with the cartesian symmetric monoidal structure.

We have an adjunction

$$(\dots)_+ : \mathbf{Top} \leftrightarrows \mathbf{Top}_* : \mathcal{F} \ ,$$

where the functor $(\dots)_+$ adds a disjoint base point, while the right adjoint $\mathcal{F}$ just forgets the base point. The left adjoint functor is **symmetric monoidal**:

$$X_+ \wedge Y_+ \cong (X \times Y)_+ \ .$$

The right-adjoint is only **lax-symmetric monoidal** (where we take $\wedge$ on the target): There is a natural map

$$\mathcal{F}(X, x) \times \mathcal{F}(Y, y) \to \mathcal{F}((X, x) \wedge (Y, y)) \ .$$

In general, it is not an isomorphism.

Of course, as a right-adjoint, the functor $\mathcal{F}$ preserves the cartesian products, i.e. it is symmetric monoidal if we take the cartesian structure on the target.

Example 2.16. For every $n \in \mathbb{N}$ we consider the sphere S^n_* as based space by distinguishing the north pole. It is actually not important which point we choose as a base point.

Problem 2.17. *Show that for every two points $x, y \in S^n$ there exists an isomorphism $(S^n, x) \cong (S^n, y)$ in* $\mathbf{Top}_*$.

For every $n \geq 1$ we have an isomorphism

$$I^n/\partial I^n \cong S^n_* \tag{4}$$

in $\mathbf{Top}_*$.

Problem 2.18. *Write down such an isomorphism explicitly.*

2.3 Suspension and loops

Let S^1_* be the circle with base point $\{1\} \in S^1 \subseteq \mathbb{C}$. We define the **suspension** functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ by

$$X \mapsto \Sigma X := S^1_* \wedge X \ .$$

We further define the loop space functor

$$\Omega : \mathbf{Top}_* \to \mathbf{Top}_*$$

by

$$\Omega X := \mathtt{Map}(S^1_*, X) \ .$$

Lemma 2.12 implies:

Corollary 2.19. *We have an adjunction*

$$\Sigma : \mathbf{Top}_* \leftrightarrows \mathbf{Top}_* : \Omega \ .$$

Lemma 2.20. *For every $n \geq 1$ we have an isomorphism*

$$S^n_* \cong \Sigma S^{n-1}_*$$

in $\mathbf{Top}_*$.

Proof. In this proof we use the isomorphisms (4).

Using $S^1_* \cong I/\partial I$ we can represent the underlying space of ΣX as

$$\Sigma X = \frac{I \times X}{\partial I \times X \cup I \times \{x\}} \ . \tag{5}$$

Remark 2.21. Using the presentation (5) we can present points in ΣX by pairs (t, x'), with $t \in I$ and $x' \in X$.

We now observe, again using Problem 2.13, that by induction

$$\Sigma(I^{n-1}/\partial I^{n-1}) \cong I/\partial I \wedge I^{n-1}/\partial I^{n-1} \cong \frac{I \times I^{n-1}}{\partial I \times I^{n-1} \cup I \times \partial I^{n-1}} \cong I^n/\partial I^n \ .$$

Indeed,

$$\partial I^n \cong \partial I \times I^{n-1} \cup I \times \partial I^{n-1} \ .$$

□

Problem 2.22. *If G is a locally compact topological group, then we have a homeomorphism*

$$LX \cong \Omega G \times G \ .$$

2.4 The homotopy category

For spaces $X, Y \in \mathbf{Top}$ we consider the following equivalence relation between morphisms $f, g \in \mathtt{Hom}_{\mathbf{Top}}(X, Y)$:

$$f \sim g \text{ - } f \text{ und } g \text{ are } \textbf{homotopic}.$$

In detail, f and g are homotopic, if there exists a map $H : I \times X \to Y$ such that $H_{|\{0\}\times X} = f$ and $H_{|\{1\}\times X} = g$.

Lemma 2.23. *Homotopy is an equivalence relation.*

Proof. Exercise. □

Remark 2.24. If X is locally compact and Hausdorff, then we can consider f, g as points in $\mathtt{Map}(X, Y)$. In this case $f \sim g$ if an only if f and g can be connected by a path of maps.□

We write $[X, Y]$ for the set of equivalence classes of maps from $X \to Y$. The composition of maps preserves the equivalence relation so that we get a well-defined composition

$$[Y, Z] \times [X, Y] \to [X, Z] \ .$$

We form the **homotopy category** $h\mathbf{Top}$ whose objects are the objects of **Top**, and whose morphism sets are given by $\mathtt{Hom}_{h\mathbf{Top}}(X, Y) := [X, Y]$. We have a natural functor

$$\mathbf{Top} \to h\mathbf{Top} \ , \quad X \mapsto X \ , \quad \mathtt{Hom}_{\mathbf{Top}}(X, Y) \ni f \mapsto [f] \in [X, Y] = \mathtt{Hom}_{h\mathbf{Top}}(X, Y) \ .$$

Problem 2.25. *Verify this assertion.*

The homotopy category has a cartesian product. In fact the cartesian product of spaces $X, Y \in h\mathbf{Top}$ is represented by the product in **Top**. In particular, the functor $\mathbf{Top} \to h\mathbf{Top}$ preserves cartesian products. A similar assertion holds true for coproducts.

Problem 2.26. *Check these assertions.*

A map $X \to Y$ is called a **homotopy equivalence**, if it represents an isomorphism in $h\mathbf{Top}$. If there exists a homotopy equivalence between X and Y, then X and Y are called **homotopy equivalent**. We write $X \simeq Y$ for the relation of being homotopy equivalence. A **homotopy type** is an isomorphism class in $h\mathbf{Top}$.

We adopt similar notions for pointed spaces. A homotopy between maps $f, g : (X, x) \to (Y, y)$ is a map

$$H : I_+ \wedge X \to Y \ ,$$

which restricts to f and g at the endpoints of I, respectively.

Let $\mathbf{C}$ be any category. A functor $F : \mathbf{Top} \to \mathbf{C}$ is called **homotopy invariant** if for any two morphisms $f, g : X \to Y$ in $\mathbf{Top}$ the relation $f \sim g$ implies that $F(f) = F(g) \in \mathtt{Hom}_{\mathbf{C}}(F(X), F(Y))$. If F is a homotopy invariant functor, then it has a canonical factorization

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "F"] \arrow[d] & \mathbf{C} \\ h\mathbf{Top} \arrow[ur, dotted] & \end{tikzcd} \ ,$$

which will usually also denoted by F.

A similar remark applies to the pointed case.

Example 2.27. The functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ is homotopy invariant and hence factorizes as

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "\pi_0"] \arrow[d] & \mathbf{Set} \\ h\mathbf{Top} \arrow[ur, dotted] & \end{tikzcd} \ .$$

Similarly, the functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Groups}$ is homotopy invariant and hence factorizes as

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, "\pi_1"] \arrow[d] & \mathbf{Groups} \\ h\mathbf{Top}_* \arrow[ur, dotted] & \end{tikzcd} \ .$$

Lemma 2.28. *If Y is a space, then the functor $\dots \wedge Y$ descends to a functor*

$$\dots \wedge Y : h\mathbf{Top}_* \to h\mathbf{Top}_* \ .$$

Similarly, if Y is locally compact and Hausdorff, then the functor $\mathtt{Map}(Y, \dots)$ descends to a functor

$$\mathtt{Map}(Y, \dots) : h\mathbf{Top}_* \to h\mathbf{Top}_* \ .$$

Proof. We show that the functors

$$\mathbf{Top}_* \stackrel{\dots \wedge Y}{\to} \mathbf{Top}_* \to h\mathbf{Top}_* \ , \quad \mathbf{Top}_* \stackrel{\mathtt{Map}(Y, \dots)}{\to} \mathbf{Top}_* \to h\mathbf{Top}_*$$

are homotopy invariant. We then get the required descended functors as canonical factorizations.

To this end it suffices to show that for homotopic maps $f, g : X \to Z$ we have $f \wedge Y \sim g \wedge Y$ and $\mathtt{Map}(Y, f) \sim \mathtt{Map}(Y, g)$, respectively. Let $H : I_+ \wedge X \to Z$ be a homotopy. Then for the first case $H \wedge Y : I_+ \wedge X \wedge Y \to Z \wedge Y$ is the required homotopy. For the second case we use the homotopy

$$I_+ \wedge \mathtt{Map}(Y, X) \to \mathtt{Map}(Y, I_+ \wedge X) \stackrel{H_*}{\to} \mathtt{Map}(Y, Z) \ .$$

Here the first map in this composition sends (t, ϕ) to the map $y \mapsto (t, \phi(y))$. $\square$

Lemma 2.29. *If the pointed space Y is locally compact Hausdorff, then we have an adjunction*

$$\dots \wedge Y : h\mathbf{Top}_* \leftrightarrows h\mathbf{Top}_* : \mathtt{Map}(Y, \dots) \ .$$

Proof. For pointed spaces X, Z we have a bijection

$$\mathtt{Hom}_{\mathbf{Top}_*}(X \wedge Y, Z) \cong \mathtt{Hom}_{\mathbf{Top}_*}(X, \mathtt{Map}(Y, Z)) \ .$$

We must show that this bijection preserves the coondition of being homotopic. Indeed, a homotopy $H \in \mathtt{Hom}_{\mathbf{Top}_*}(I_+ \wedge X \wedge Y, Z)$ corresponds bijectively to a homotopy $\tilde{H} \in \mathtt{Hom}_{\mathbf{Top}_*}(I_+ \wedge X, \mathtt{Map}(Y, Z))$. Therefore we have a natural bijection $[X \times Y, Z] \cong [X, \mathtt{Map}(Y, Z)]$. $\square$

Note that similar statements hold true for the unpointed case.

Corollary 2.30. *We have an adjunction*

$$\Sigma : h\mathbf{Top}_* \leftrightarrows h\mathbf{Top}_* : \Omega \ .$$

2.5 H and co-H-spaces

The main problem of homotopy theory is the calculation of the set $[X, Y]$ for spaces $X, Y \in \mathbf{Top}$, or the similar problem in the pointed case. In general, these sets are extremely difficult to describe and the presence of some algebraic structures is very helpful. In this subsection we discuss structures on X or Y which induce group structures on $[X, Y]$.

Example 2.31. If H is a topological group, then the set $[X, H]$ has the structure of a group with multiplication induced by pointwise multiplication of representatives. In order to define a group structure on $[X, H]$ we do not really need a topological group structure on H, but a much weaker structure which we will discuss next.

Definition 2.32. *An H-space is a pointed space $(H, *)$ together with a multiplication map $\mu : H \times H \to H$ which turns $((H, *), [\mu])$ into a monoid object in $h\mathbf{Top}_*$.*

Remark 2.33. In detail, this means that we have the associativity relation

$$\mu(\mathtt{id} \times \mu) \sim \mu(\mu \times \mathtt{id})$$

and the unit relation

$$\mu(*, \dots) \sim \mathtt{id}_H \sim \mu(\dots, *) \ .$$

A pointed space H represents a functor

$$[\dots, H] : h\mathbf{Top}_*^{op} \to \mathbf{Set} \ , \quad X \mapsto [X, H] \ .$$

Similarly, an H-space H represents a functor

$$[\dots, H] : h\mathbf{Top}_*^{op} \to \mathbf{Monoids} \ .$$

The underlying set-valued functor is defined as above. The multiplication is given by

$$[X, H] \times [X, H] \cong [X, H \times H] \stackrel{\mu_*}{\to} [XH] \ .$$

Finally, the unit of the monoid structure on $[X, H]$ is induced by the base point $* = [X, *] \to [X, H]$ of the H-space.

An H-space is called **group like** if the functor $[\dots, H]$ takes values in groups. It is called **commutative**, if $[\dots, H]$ takes values in commutative monoids.

Example 2.34. A topological group is naturally a group-like H-space.

A morphism between H-spaces is a map between pointed spaces which induces a natural transformation of monoid-valued functors. We get a category of H-spaces. In particular we have a notion of an **isomorphism** of H-spaces. We say that two H-spaces a **equivalent** if the are isomorphic monoid objects in $h\mathbf{Top}$.

Problem 2.35. *Verify the following assertions: Up to isomorphism, to give an H-space structure on a space H is the same as to give a group structure on the functor $[\dots, H]$.*

Lemma 2.36. *For $X \in \mathbf{Top}_*$ the loop space ΩX is a group-like H-space with the operation given by concatenation of loops.*

Proof. Exercise. $\square$

For a pointed space X we can consider the double loop space Ω^2X. It has two H-space structures $+$ and $*$. The first one is the structure on the loop space $\Omega(\Omega X)$. The second one is given by

$$\Omega^2(X) \times \Omega^2(X) \cong \Omega(\Omega(X)) \times \Omega(\Omega(X)) \stackrel{\Omega\mu}{\to} \Omega^2(X) \ .$$

For the first equivalence we use that Ω commutes with products since it is a right-adjoint.

Lemma 2.37. *The two H-space structures are isomorphic. Furthermore, they are both abelian.*

Proof. We use the identification $\Omega^2X = \mathtt{Map}(I^2/\partial I^2, X)$ and represents elements in Ω^2X by maps $I^2 \to X$ mapping the boundary to the base point. Given elements $a, b, c, d \in \Omega^2X$ we can define a map $I^2/\partial I^2 \to X$ described by

$$\begin{array}{|c|c|} \hline b & d \\ \hline a & c \\ \hline \end{array} \ .$$

The two H-spaces structures $+$ and $*$ are the horizontal and vertical composition. We immediately read off:

$$(a * b) + (c * d) = (a + c) * (b + d) \ . \tag{6}$$

The **Eckmann-Hilton** argument gives $+ = *$, and that both structures a commutative. $\square$

Remark 2.38. On could ask if every group-like H-space is equivalent to a loop space. This is not the case. The H-space structure of a loop space is not just associative up to homotopy, but these homotopies satisfy an infinite chain of higher coherence conditions.

If H is an H-space with these additional higher homotopies given, then we can construct a so-called a classifying space BH together with a map of H-spaces $H \to \Omega BH$. If H is group-like then this morphism is an equivalence of H-spaces. $\square$

Let X, Y be a pointed spaces. By Corollary 2.30 we have the natural bijection

$$[\Sigma X, Y] \cong [X, \Omega Y] \ .$$

Therefore the suspension ΣX represents a functor

$$[\Sigma X, \dots] : \mathbf{Top}_* \to \mathbf{Groups} \ .$$

The multiplication must be (see Problem 2.35 for the dual statement) corepresented by a comultiplication map

$$\delta : \Sigma X \to \Sigma X \vee \Sigma X \ .$$

Indeed, we have a map

$$[\Sigma X \vee \Sigma X, \Sigma X \vee \Sigma X] \cong [\Sigma X, \Sigma X \vee \Sigma X] \times [\Sigma X, \Sigma X \vee \Sigma X] \to [\Sigma X, \Sigma X \vee \Sigma X] \ ,$$

where the first given by the universal property of the coproduct $\vee$ in pointed spaces, and the second is the multiplication. The comultiplication map δ is the image of $\mathtt{id}_{\Sigma X \vee \Sigma X}$ under this composition. This is a co-H-space structure.

Definition 2.39. *A co-H-space is a pointed space $(H, *)$ together with a comultiplication map $\delta : H \to H \vee H$ which turns $((H, *), [\delta])$ into a comonoid object in* $\mathbf{Top}_*$.

Remark 2.40. In detail, this means that we have the associativity relation

$$(\mathtt{id} \vee \delta)\delta \sim (\delta \vee \mathtt{id})\delta \ ,$$

the unit relation

$$\mathtt{pr}_0 \circ \delta \sim \mathtt{id}_H \sim \mathtt{pr}_2 \circ \delta \ .$$

A pointed space H corepresents a functor

$$[H, \dots] : h\mathbf{Top}_* \to \mathbf{Set}_* \ , X \mapsto [H, X] \ .$$

Similarly, a co-H-space corepresents a functor

$$[H, \dots] : h\mathbf{Top}_* \to \mathbf{Monoids} \ .$$

The operation on $[H, X]$ is given by

$$[H, X] \times [H, X] \cong [H \vee H, X] \xrightarrow{\delta^*} [H, X] \ .$$

A morphism between co-H-spaces is a map of pointed spaces which induces a morphism of comonoids in the homotopy category. We again get a category of co-H-spaces and a corresponding notion of isomorphism. Similarly as in Problem 2.35, up to isomorphism, a co-H-space structure on a pointed spaces H is determined by a monoid structure on the functor $[H, \dots] : h\mathbf{Top}_* \to \mathbf{Monoids}$.

Example 2.41. For a pointed space we already know that ΣX has a co-h-space which is uniquely determined up to isomorphism

We represent points in ΣX by pairs $(t, x) \in I \times X$. Explicitly, the coproduct is given by

$$(t, x) \mapsto \begin{cases} (2t, x) & \text{left copy of the wedge} \quad t \in [0, 1/2) \\ (2t - 1, x) & \text{right copy of the wedge} \quad t \in [1/2, 1] \end{cases}$$

A co-H-space is called **cogroup like** if the functor $[H, \dots]$ takes values in groups. It is called **cocommutative**, if $[H, \dots]$ takes values in commutative monoids.

Example 2.42. For a pointed space X the co-H-space ΣX is cogroup like.

For a space X we can consider the double suspension $\Sigma^2 X$. It has two co-H-space structures.

Lemma 2.43. *The two co-H-space structures are equivalent. Furthermore, they are both abelian.*

Proof. This is similar as Lemma 2.37. □

Definition 2.44. *Let $(X, x) \in \mathbf{Top}_*$. For $n \in \mathbb{N}$ we define the set*

$$\pi_n(X, x) \cong [S^n_*, (X, x)] \ .$$

with the following structures:

1. *$n = 0$: $\pi_0(X, x)$ is a pointed set with base point represented by the constant map.*
2. *$n = 1$: $\pi_1(X, x)$ is a group with structure induced via*

$$[S^1_*, X] \cong [\Sigma S^0_*, X] \ .$$

   and the co-H-space structure on ΣS^0_*.

3. *$n \geq 2$: $\pi_n(X, x)$ is an abelian group with structure induced via*

$$[S^n_*, X] \cong [\Sigma^2 S^{n-2}_*, X]$$

   and the co-H-space structure on S^{n-2}_*.

For $i \geq 2$ the abelian groups $\pi_i(X, x)$ are called the ***higher homotopy groups*** *of X.*

We get induced functors

$$\pi_i : h\mathbf{Top}_* \to \begin{cases} \mathbf{Set}_* & i = 0 \\ \mathbf{Groups} & i = 1 \\ \mathbf{Ab} & i \geq 2 \end{cases} .$$

Note that by Lemma 2.20 and Corollary 2.19 we have

$$\pi_n(X, x) \cong [S^n_*, X] \cong [\Sigma^n S^0_*, X] \cong [S^0_*, \Omega^n X]$$

and that we can induce the group structures from the H-space structures on $\Omega^n X$.

Definition 2.45. *A map $f : X \to Y$ between spaces is called a **weak equivalence** if it induces isomorphisms after applying $\pi_i(X, x) \to \pi_i(Y, f(x))$ for all $i \in \mathbb{N}$ and $x \in X$.*

If $X \to Y$ is a homotopy equivalence, then clearly it is a weak equivalence.

Problem 2.46. *Check this assertion.*

Remark 2.47. The converse is not true. To this end we introduce a class of nice spaces called CW-complexes. It will turn out that if we restrict to the class of CW-complexes, then weak homotopy equivalence is a homotopy equivalence.

3 Cofibre sequences

3.1 The mapping cone

Let $f : X \to Y$ a map between spaces. We define the quotient $Y/f(X)$ by the push-out

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Y/f(X) \end{tikzcd} .$$

In general, the homotopy type of $Y/f(X)$ is not an invariant of the homotopy class of the map f.

Example 3.1. The inclusion of $f_0 : S^0 \to [0, 1]$ as boundary is homotopic to the constant map $f_1 : S^0 \to [0, 1]$ with value $\{0\}$. But $[0, 1]/f_0(S^0) \cong S^1$ while $[0, 1]/f_1(S^0) \cong [0, 1]$ and $S^1 \not\simeq [0, 1]$.

The mapping cone $C(f)$ of f is a homotopy invariant replacement of the quotient $Y/f(X)$. In good cases it will be homotopy equivalent to this space.

Definition 3.2. *The **mapping cone** $C(f)$ of f is defined as the push-out*

$$\begin{tikzcd} \{0\}\times X\sqcup\{1\}\times X \arrow[r, "p\sqcup f\circ \mathtt{pr}_X"] \arrow[d] & *\sqcup Y \arrow[d, "i"] \\ {[0,1]}\times X \arrow[r, "j"] & C(f) \end{tikzcd} .$$

Here $p:\{0\}\times X\to *$ is the canonical map. The mapping cone has a base point given by the image of $*$. It comes with an inclusion $i:Y\to C(f)$.

Remark 3.3. A description of the mapping cone of f in words is as follows. One first attaches Y to $[0,1]\times X$ by identifying the points $(1,x)$, $x\in X$ with $f(x)\in Y$. The resulting space is also called the **mapping cylinder** $Z(f)$ of f. Then one contracts the subset $\{0\}\times X\subset Z(f)$ to the base point.

In the pointed category we adopt a similar definition using the corresponding notions of products and coproducts. Using the smash product we get a simplified formula

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[d, "i"] \\ {[0,1]}\wedge X \arrow[r, "j"] & C(f) \end{tikzcd} .$$

Here $0\in[0,1]$ is the base point.

Example 3.4. The mapping cone of the identity $C(X):=C(\mathtt{id}_X)$ (in the unpointed case) is homeomorphic to $[0,1]\times X/\{0\}\times X$. It is a contractible space called the **cone over** X. A contraction to the base point is given by

$$[0,1]\times C(X)\ni(s,[t,x])\mapsto[(1-s)t,x]\in C(X)\ .$$

In terms of the cone over X we can describe the mapping cone of a map $f:X\to Y$ as the push-out

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[d, "i"] \\ C(X) \arrow[r] & C(f) \end{tikzcd} .$$

We use this description in order to denote points in $C(f)$ by $[t,x]\in C(X)$ with $t\in I$ and $x\in X$ and $y\in Y$.

Example 3.5. In the pointed case the mapping cone of the inclusion of the base point $* \to X$ is homeomorphic to X.

Remark 3.6. For a map $f : X \to Y$ between based spaces the quotient $Y/f(X)$ is characterized by the following universal property: For every based test space T to give a map $\phi : Y/f(X) \to T$ is the same as to give a map $\tilde{\phi} : Y \to T$ such that the composition $h \circ f$ is the constant map.

The mapping cone has a similar universal property: For every test space T to give a map $\phi : C(f) \to T$ is the same as to give a map $\tilde{\phi} : Y \to T$ together with a homotopy $h : [0,1] \times X \to T$ from the constant map to the composition $f \circ \tilde{\phi}$. For an argument see also the proof of Lemma 3.12 below. □

Let $f : X \to Y$ be an embedding of a subspace. We consider $Y/X := Y/f(X)$ as a based space. Then we have a natural map of pointed spaces

$$p : C(f) \to Y/X \ , \quad \begin{cases} [t,x] \mapsto * & \text{on } C(X) \\ y \mapsto [y] & \text{on } Y \end{cases} \tag{7}$$

We ask when this map is a homotopy equivalence.

Definition 3.7. *We call a subspace $f : X \hookrightarrow Y$ a **neighbourhood deformation retract (NDR)** if there exist the following data:*

1. *a neighbourhood $\tilde{f} : U \hookrightarrow Y$ together with a retraction $r : U \to X$ (i.e. $r_{|X} = \mathtt{id}_X$),*
2. *a homotopy $H : I \times U \to U$ from $f \circ r$ to $\mathtt{id}_U$,*
3. *a function $\chi \in C(Y)$ such that $\chi_{|X} = 0$ and $\chi_{|Y/U} = 1$.*

Lemma 3.8. *If $f : X \hookrightarrow Y$ is a NDR , then the canonical projection $p : C(f) \to Y/X$ is a homotopy equivalence.*

Proof. We define a homotopy inverse $q : Y/X \to C(f)$ as follows:

$$q([y]) := \begin{cases} [2\chi(y), r(y)] \ \text{ in } C(X) & y \in U \ , \chi(y) < 1/2 \\ H(2\chi(y) - 1, y) \ \text{ in } Y & y \in U \ , \chi(y) \geq 1/2 \\ y \ \text{ in } Y & y \in Y \setminus U \end{cases}$$

Then we have

$$p \circ q([y]) = \begin{cases} [y] & y \in Y \setminus U \\ [H(2\chi(y) - 1, y)] & y \in U, \chi(y) \geq 1/2 \\ * & y \in U \ , \chi(y) < 1/2 \end{cases} .$$

A homotopy from the identity to $p \circ q$ is given by

$$(s, [y]) \mapsto \begin{cases} [y] & y \in Y \setminus U \\ [H(s(2\chi(y) - 1) + (1 - s), y)] & y \in U \ , \end{cases} ,$$

where we define $H(s, y) := r(y)$ for $s < 0$.

Here is the description of the homotopy from $q \circ p$ to the identity. We distinguish four types of points.

1. y for $y \in Y \setminus U$: The homotopy fixes these points.
2. y for $y \in U$ and $\chi(y) \geq 1/2$: The homotopy is the path $s \mapsto H((1-s)(2\chi(y)-1) + s, y)$ in Y.
3. y for $y \in U$ and $\chi(y) < 1/2$: The homotopy is the concatenation of the path $s \mapsto [(1-s)2\chi(y)+s, r(y)]$ in $C(X)$ and the path $s \mapsto H(s, y)$ in Y. Here we rescale the paths such that for the first part we need the time $2(1/2 - \chi(y))$ and the rest of the time is used for the second part.
4. $[t, x]$ in $C(X)$: The homotopy is the path $[ts, x]$ in $C(X)$.

Problem 3.9. *Show that this really defines the desired homotopy.*

□

Example 3.10. Let X be a space and $Y := X \cup_{S^{n-1}} D^n$ be obtained by **attaching an n-cell** along the boundary. More precisely, Y is defined as the push-out

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} .$$

Then the embedding $X \to Y$ is an NDR, see Definition 3.7.

Indeed, X is a deformation retract of the neighbourhood $U := X \cup_{S^{n-1}} (D^n \setminus \{0\})$. The retraction is induced by the map

$$D^{n-1} \setminus \{0\} \to S^{n-1} \ , \quad u \mapsto \frac{u}{\|u\|} \ .$$

The homotopy is induced by $[0,1] \times (D^n \setminus \{0\}) \ni (t, u) \mapsto tu + (1-t)\frac{u}{\|u\|}$.

The function $\chi : Y \to [0, 1]$ is induced by the universal property of the push-out by the functions $u \mapsto 1 - \|u\|$ on D^n and 0 on X.

Note that $Y/X \cong D^n/S^{n-1} \cong S^n$. So in this case $C(X \to Y) \simeq S^n$.

3.2 The mapping cone sequence

We call

$$X \xrightarrow{f} Y \xrightarrow{i} C(f)$$

the **mapping cone sequence** of f.

Definition 3.11. *A sequence of maps between pointed sets*

$$(A, a) \xrightarrow{\phi} (B, b) \xrightarrow{\psi} (C, c)$$

is called ***exact*** *if* $\psi^{-1}(c) = \phi(A)$.

Let (Z, z) be a pointed space. For any space X the set $[X, Z]$ of homotopy classes of maps from X to Z is pointed by the constant map to the base point.

Lemma 3.12. *The mapping cone sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} C(f)$$

induces an exact sequence of maps between pointed sets

$$[(C(f), *), (Z, z)] \xrightarrow{i^*} [Y, Z] \xrightarrow{f^*} [X, Z] \ .$$

Proof. Let $[u] \in [Y, Z]$. The condition that $f^*[u] \in [X, Z]$ is the base point is equivalent to the existence of map $H : [0, 1] \times X \to Z$ such that $H_{|\{1\}\times X} = u \circ f$ and $H_{\{0\}\times X}$ is the constant map to z. The datum of H and u together, by the universal property of the push-out, is equivalent to a map of based spaces $h : (C(f), *) \to Z$ such that $h \circ i = u$ and $h \circ j = H$. So in particular $[u] = i^*[h]$.

On the other hand, if $[h] \in [(C(f), *), (Z, z)]$, then $H := h \circ j : [0, 1] \times X \to Z$ is a homotopy from the constant map with value z to the composition $h \circ i \circ f$. This implies that $f^*i^*[h]$ is the base point of $[X, Z]$.

□

There is an analog of this Lemma in the based case. Assume that $f : (X, x) \to (Y, y)$ is a map of based spaces.

Lemma 3.13. *The mapping cone sequence*

$$(X, x) \xrightarrow{f} (Y, x) \xrightarrow{i} (C(f), *)$$

induces an exact sequence of maps between pointed sets

$$[(C(f), *), (Z, z)] \xrightarrow{i^*} [(Y, y), (Z, z)] \xrightarrow{f^*} [(X, x), (Z, z)] \ .$$

□

3.3 Functoriality properties of the mapping cone

We consider the category $\Delta^1 := (\bullet \to \bullet)$. For any category $\mathbf{C}$ we let

$$\mathbf{C}^{\Delta^1} := \mathbf{Fun}(\Delta^1, \mathbf{C})$$

be the **category of morphisms** in $\mathbf{C}$. In explicit terms, its objects are morphisms $f : X \to Y$ between objects of $\mathbf{C}$, and its morphisms

$$(\phi, \psi) : (f : X \to Y) \to (f' : X' \to Y')$$

are commutative squares

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} .$$

Let $(f : X \to Y) \in \mathbf{Top}^{\Delta^1}$ be a morphism in $\mathbf{Top}$. We consider the map $i : Y \to C(f)$ as an object of $\mathbf{Top}^{\Delta^1}$.

Lemma 3.14. *The mapping cone construction can be considered as a functors*

$$C : \mathbf{Top}^{\Delta^1} \to \mathbf{Top}^{\Delta^1} \ , \quad C : \mathbf{Top}_*^{\Delta^1} \to \mathbf{Top}_*^{\Delta^1} \ .$$

Proof. Exercise. □

In the following we consider the unpointed and pointed cases simultaneously. Let Z be a space. If $f : X \to Y$ is a morphism, then we can form $Z \times f : Z \times X \to Z \times Y$, or $Z \wedge f : Z \wedge X \to Z \wedge Y$ in the pointed case.

Lemma 3.15. *There a natural isomorphisms*

$$C(Z \times f) \cong Z_+ \wedge C(f) \ , \quad C(Z \wedge f) \cong Z \wedge C(f) \ .$$

Proof. Exercise. □

There is a natural notion of a homotopy between morphisms in $\mathbf{Top}^{\Delta^1}$ or $\mathbf{Top}_*^{\Delta^1}$.

Corollary 3.16. *On morphisms the cone functor preserves homotopies.*

We consider again a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd} .$$

We say, that it **commutes up to a distinguished homotopy** H if we are given a homotopy $H : f' \circ \phi \sim \psi \circ f$. We consider the datum (ϕ, ψ, H) as a sort of **generalized morphism** from f to f'.

For a commuting square we can take the constant homotopy.

Lemma 3.17. *Assume*

$$(\phi, \psi, H) : \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

is a generalized homomorphism. Then we get a canonical commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "\psi"] & C(f) \arrow[d, "C(\phi,\psi,H)"] \\ Y' \arrow[r] & C(f') \end{tikzcd} . \tag{8}$$

Proof. Let $H : [0, 1] \times X \to Y'$ be the homotopy $f' \circ \phi \sim \psi \circ f$. We define a map

$$C(\phi, \psi, H) : C(f) \to C(f')$$

by

$$C(X) \ni [t, x] \mapsto \begin{cases} [2t, \phi(x)] & \text{in } C(X') \quad t \in [0, 1/2] \\ H(2t-1) & \text{in } Y' \quad\quad t \in (1/2, 1] \end{cases} \in C(f')$$

and

$$Y \ni y \mapsto \psi(y) \ \text{ in } Y' .$$

□

Lemma 3.18. *The functor map $(\phi, \psi, H) \mapsto C(\phi, \psi, H)$ preserves homotopies.*

Proof. Exercise. □

There is a natural way to **compose generalized morphisms** $(\phi, \psi, H) : f \to f'$ and $(\phi', \psi', H') : f' \to f''$ between maps by putting one square on top of the other and concatenating (we use the symbol $\sharp$) the homotopies to the new homotopy

$$(H' \circ \phi)\sharp(\psi' \circ H)$$

Lemma 3.19. *The maps $C(\phi, \psi, H) \circ C(\phi', \psi', H')$ and $C(\phi \circ \phi', \psi \circ \psi', H \circ H')$ are homotopic.*

Proof. Exercise. □

A generalized morphism

$$(\phi, \psi, H) : (f : X \to Y) \to (f' : X' \to Y')$$

is a homotopy equivalence if there exist a morphism

$$(\phi', \psi', H') : X' \to Y') \to (f : X \to Y)$$

such that the two obvious compositions are homotopic to the identity morphisms.

Corollary 3.20. *If (ϕ, ψ, H) is a homotopy equivalence, then $C(\phi, \psi, H)$ is a homotopy equivalence.*

The following Lemma shows that for a generalized morphism (ϕ, ψ, H) the condition of beeing a homotopy equivalence only depends on ϕ and ψ, but not on the homotopy H.

Lemma 3.21. *Consider a square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

which commutes up to the distinguished homotopy H and assume that ϕ and ψ are homotopy equivalences. Then $C(\phi, \psi, H)$ is a homotopy equivalence.

Proof. We can choose homotopy inverses ϕ' and ψ'. Then the square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\phi'"] & Y' \arrow[d, "\psi'"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes up to some homotopy H'. In order to define H' we first choose homotopies $\mathtt{id}_Y \sim \psi' \circ \psi$ and $\phi \circ \phi' \sim \mathtt{id}_X$. Then we get the desired homotopy H' as the following concatenation of homotopies between maps $X' \to Y$

$$f \circ \phi' \sim \psi' \circ \psi \circ f \circ \phi' \overset{H}{\sim} \psi' \circ f' \circ \phi \circ \phi' \sim \psi' \circ f' \ .$$

Problem 3.22. *We leave it as an exercise to check that the square* (ϕ', ψ', H') *is a homotopy inverse of* (ϕ, ψ, H).

$\square$

3.4 The long mapping cone sequence

We now restrict to the pointed case. If we consider the mapping cone construction C as an endofunctor of $\mathbf{Top}_*^{\Delta^1}$ then we can iterate it. Given a map between pointed spaces $f : X \to Y$ we define $f^{(k)} := C^k(f)$. We get a sequence of pointed spaces

$$\mathcal{P}(f) : X \overset{f^{(0)}}{\to} Y \overset{f^{(1)}}{\to} C(f^{(0)}) \overset{f^{(2)}}{\to} C(f^{(1)}) \overset{f^{(3)}}{\to} \dots \ .$$

It will also be called the long mapping cone sequence or **Puppe sequence**. For a pointed space (Z, z) we get a long exact sequence of sets

$$[\mathcal{P}(f), (Z, z)] \ .$$

At a first glance the iterated mapping cones seem to become increasingly more complicated. But the opposite is the case as we shall see now.

We can extend the mapping cone sequence of f by

$$X \overset{f}{\to} Y \overset{i}{\to} C(f) \overset{k}{\to} C(i) \ .$$

There exists a natural map

$$q : C(f) \to \Sigma X$$

which is induced by the map of push-out diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} \\ C(X) & \longrightarrow & C(f) \end{array} \quad \to \quad \begin{array}{ccc} X & \xrightarrow{f} & * \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} \\ C(X) & \xrightarrow{j} & \Sigma X \end{array} \ .$$

Lemma 3.23. *There exists a homotopy equivalence $a : C(i) \xrightarrow{\sim} \Sigma X$ such that*

$$\begin{tikzcd} C(f) \arrow[r, "k"] \arrow[d, equal] & C(i) \arrow[d, "a"] \\ C(f) \arrow[r, "q"] & \Sigma X \end{tikzcd}$$

commutes.

Proof. In order to understand $C(i)$ we recall its definition

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "1"] & C(f) \arrow[d, "k"] \\ C(Y) \arrow[r] & C(i) \end{tikzcd} .$$

A good way to imagine $C(i)$ is as the space obtained by glueing a cone over X with a cone over Y along their bases using the map f, i.e. $C(i) \cong C(X) \cup_{f:X\to Y} C(Y)$. We call these subspaces the left and right summands.

We write points in $C(i)$ as $[t, x]$ (in the left summand) or $[t, y]$ (in the right summand) with $t \in [0, 1]$ and $x \in X$ and $y \in Y$ such that $[1, x] = [1, f(x)]$.

We define the map

$$a : C(i) \to \Sigma X$$

such that it is the projection $C(X) \to \Sigma X$ on the left summand and the projection $C(Y) \to *$ on the right summand.

An inverse map is given by

$$b : \Sigma X \to C(i) \ , \quad [t, x] \mapsto \left\{ \begin{array}{ccc} [2t, x] & \text{left summand} & t \in [0, 1/2] \\ [2 - 2t, f(x)] & \text{right summand} & t \in [1/2, 1] \end{array} \right. . \tag{9}$$

Then the composition $a \circ b : \Sigma X \to \Sigma X$ is given by

$$a \circ b : \Sigma X \ni [t, x] \mapsto \left\{ \begin{array}{cc} [2t, x] & t \in [0, 1/2] \\ * & t \in [1/2, 1] \end{array} \right. .$$

Hence $a \circ b$ is homotopic to the identity. We describe how the homotopy moves the point $[t, x] \in \Sigma X$. We consider two cases:

1. $t \leq 1/2$: The homotopy applied to $[t, x]$ is the path $[(1 - s)2t + st, x]$.
2. $t \geq 1/2$: The homotopy is the path $[st + (1 - s), x]$.

We further see that the composition $b \circ a : C(i) \to C(i)$ is given by

$$[t, x] \to \left\{ \begin{array}{cc} [2t, x] & t \in [0, 1/2] \\ [2 - 2t, f(x)] & t \in (1/2, 1] \end{array} \right. , \quad [t, y] \mapsto * .$$

We now describe the homotopy to the identity

1. $[t,x]$ for $t \le 1/2$: The homotopy is the path $[(1-s)2t+st, x]$.
2. $[t,x]$ for $t \ge 1/2$: The homotopy is the concatenation of the paths $s \mapsto [(1-s)(2-2t), f(x)]$ and $s \mapsto [(1-s)+st, x]$. Here we rescale the path such that the first path is run through in the time $(2-2t)$ and the rest of the time is used for the second piece.
3. $[t,y]$: The homotopy is the path $[(1-s)+st, y]$.

$\square$

We now get a natural commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f^{(0)}"] \arrow[d, equal] & Y \arrow[r, "f^{(1)}"] \arrow[d, equal] & C(f^{(0)}) \arrow[r, "f^{(2)}"] \arrow[d, equal] & C(f^{(1)}) \arrow[r, "f^{(3)}"] \arrow[d, "\simeq"', "a"] & C(f^{(2)}) \arrow[d, "\simeq"'] \arrow[dd, bend left, "\simeq\, a^{(1)}"] \\
X \arrow[r, "f"] & Y \arrow[r, "i"] & C(f) \arrow[r, "q"] & \Sigma X \arrow[r] \arrow[d, equal] & C(q) \\
& & & \Sigma X \arrow[r, "?"] & \Sigma Y
\end{tikzcd} .$$

The square $(4,0)$ is obtained from the functoriality of $C(f)$ applied to the square $(3,0)$.

For a space X we consider the involution

$$- : \Sigma X \to \Sigma X \ , \qquad [t,x] \mapsto [1-t,x] \ .$$

For a map $f : X \to Y$ we write $-\Sigma f$ for the composition of Σ with this involution.

Lemma 3.24. *Up to homotopy we have* $? \sim -\Sigma f$.

Proof. We identify the map ? using the homotopy inverse b of a. In view of (9) the composition $a^{(1)} \circ f^{(3)} \circ b$ is given by

$$[t,x] \mapsto \begin{cases} * & [t \in 0, 1/2] \\ [2-2t, y] & t \in (1/2, 1] \end{cases} .$$

$\square$

Corollary 3.25. *We get equivalences*

$$\Sigma C(f) \simeq C(-\Sigma f) \ .$$

Furthermore, the long mapping cone sequence is equivalent to

$$X \stackrel{f}{\to} Y \stackrel{i}{\to} C(f) \stackrel{q}{\to} \Sigma X \stackrel{-\Sigma f}{\to} \Sigma Y \stackrel{-\Sigma i}{\to} \Sigma C(f) \stackrel{-\Sigma q}{\to} \Sigma^2 X \stackrel{\Sigma^2 f}{\to} \Sigma^2 Y \stackrel{\Sigma^2 i}{\to} \Sigma^2 C(f) \to \dots$$

Corollary 3.26. *If Z is a pointed space, then we get a long exact sequence of pointed sets*

$$\begin{tikzcd}
 & \cdots & . \arrow[dll] \\
{[\Sigma^2 C(f), Z]} \arrow[r] & {[\Sigma^2 Y]} \arrow[r] & {[\Sigma^2 X, Z]} \arrow[dll] \\
{[\Sigma C(f), Z]} \arrow[r] & {[\Sigma Y]} \arrow[r] & {[\Sigma X, Z]} \arrow[dll] \\
{[C(f), Z]} \arrow[r] & {[Y, Z]} \arrow[r] & {[X, Z]}
\end{tikzcd}$$

From the second line on we have an exact sequence of groups which are abelian from the third line on. The group $[\Sigma X]$ acts simply-transitively on the fibre of the map $[C(f), Z] \to [Y, Z]$.

4 Cohomology

4.1 Spectra

A **spectrum** $\mathbf{E} = ((E_i)_{i\in\mathbb{Z}}, (\sigma_i)_{i\in\mathbb{Z}})$ is a sequence of pointed spaces $(E_i)_{i\in\mathbb{Z}}$ together with structure maps

$$\sigma_i : E_i \to \Omega E_{i+1}$$

for all $i \in \mathbb{Z}$ which are assumed to be homotopy equivalences.

Remark 4.1. At the moment we do not have the means to construct an example of a spectrum. So we shall just assume that spectra exist. As we will see shortly, a spectrum gives rise to a cohomology theory. If one represents a cohomology theory by a spectrum, then the basic structures follow in an easy manner. As the reader might know, there are other ways to construct cohomology theories. Here the verification of the properties are more involved. So by the **principle of preservation of difficulty** the construction of a spectrum should be a non-trivial matter.

We fix a spectrum $\mathbf{E}$. If X is a pointed space, then $[X, E_i]$ is a pointed set. The structure of the spectrum $\mathbf{E}$ provides isomorphisms of pointed sets

$$[X, E_i] \cong [X, \Omega E_{i+1}] \cong [X, \Omega^2 E_{i+2}] \ .$$

Using these isomorphisms we equip the pointed set $[X, E_i]$ with an abelian group structure.

Definition 4.2. *We define the* ***(reduced) cohomology*** *of a pointed space X with coefficients in $\mathbf{E}$ to be the $\mathbb{Z}$-graded abelian group $H^*(X; \mathbf{E})$ whose degree-i-component is*

$$H^i(X; \mathbf{E}) := [X, E_i] \ .$$

If X un-pointed, then define the cohomology of X with coefficients in $\mathbf{E}$ by

$$H^*(X;\mathbf{E}) := H^*(X_+;\mathbf{E}) \ .$$

If $f : X \to Y$ is a map of spaces, then the induced map

$$f^* : H^*(Y;\mathbf{E}) \to H^*(X;\mathbf{E})$$

is a homomorphism of $\mathbb{Z}$-graded abelian groups. The homomorphism f^* only depends on the homotopy class of f, and for composeable maps we have the relation $(g \circ f)^* = f^* \circ g^*$. We let $\mathbf{Ab}_{\mathbb{Z}-gr}$ denote the category of $\mathbb{Z}$-graded abelian groups.

Corollary 4.3. *A spectrum $\mathbf{E}$ induces functors*

$$H^*(\dots;\mathbf{E}) : h\mathbf{Top}_*^{op} \to \mathbf{Ab}_{\mathbb{Z}-gr} \ , \quad H^*(\dots;\mathbf{E}) : h\mathbf{Top}^{op} \to \mathbf{Ab}_{\mathbb{Z}-gr}$$

Lemma 4.4. *For every pointed space X and integer $i \in \mathbb{Z}$ we have the natural* ***suspension isomorphism***

$$H^{i+1}(\Sigma X;\mathbf{E}) \cong H^i(X;\mathbf{E}) \ .$$

Proof. The suspension isomorphism is given by the chain of natural isomorphisms

$$H^{i+1}(\Sigma X;\mathbf{E}) \cong [\Sigma X, E_{i+1}] \cong [X, \Omega E_{i+1}] \cong [X, E_i] = H^i(X;\mathbf{E}) \ .$$

□

The cohomology with coefficients satisfies the **wegde axiom**:

Lemma 4.5. *Assume that $X \cong \bigvee_{\alpha\in I} X_\alpha$. Then for every $i \in \mathbb{Z}$ we have an isomorphism*

$$H^i(X;\mathbf{E}) \to \prod_{\alpha\in I} H^i(X_\alpha;\mathbf{E})$$

induced by the natural map.

Proof. The relation $X \simeq \bigvee_{\alpha\in I} X_\alpha$ holds true in the homotopy category. Mapping out turns coproducts into products. □

Definition 4.6. *For a map $f : X \to Y$ between pointed spaces we define the cohomology of f by*

$$H^i(f;\mathbf{E}) := H^i(C(f);\mathbf{E}) \ .$$

If f is the inclusion of a subspace, then one often uses the notation

$$H^i(Y, X;\mathbf{E}) := H^i(f;\mathbf{E})$$

and calls this the ***cohomology of the pair*** *(Y, X).*

Lemma 4.7. *For a map $f : X \to Y$ between pointed spaces we have a* ***long exact cohomology sequence***

$$\to H^{i-1}(X;\mathbf{E}) \xrightarrow{q^*} H^i(f;\mathbf{E}) \xrightarrow{i^*} H^i(Y;\mathbf{E}) \xrightarrow{f^*} H^i(X;\mathbf{E}) \to \dots$$

Proof. By Corollary 3.26 we have an exact sequence

$$\to H^i(\Sigma X,\mathbf{E}) \xrightarrow{q^*} H^i(C(f),\mathbf{E}) \xrightarrow{i^*} H^i(Y,\mathbf{E}) \xrightarrow{f^*} H^i(X;\mathbf{E}) \to \dots$$

and now use the suspension isomorphism at the first entry. □

The $\mathbb{Z}$-graded abelian group

$$E^* := H^*(S^0_*;\mathbf{E})$$

is called the coefficients of the cohomology theory. Note that

$$H^i(S^0_*,\mathbf{E}) \cong H^{i+n}(S^n_*;\mathbf{E})$$

for all $n \in \mathbb{N}$.

Remark 4.8. In the **axiomatic approach to reduced cohomology theories** these properties are the axioms. Here a reduced cohomology theory is a functor

$$h^* : \mathbf{Top}_* \to \mathbf{Ab}_{\mathbb{Z}-gr}$$

with the following properties and additional structures:

1. It is homotopy invariant.
2. It satisfies the wedge axiom.
3. It has the additional structure of natural suspension isomorphisms $h^{*+1}(\Sigma X) \cong h^*(X)$.
4. For every $f : X \to Y$ the sequence $h^*(C(f)) \to h^*(Y) \to h^*(X)$ is exact in the middle.

The groups $h^*(S^0_*)$ are called the coefficients of the cohomology theory.

Example 4.9. If $(\mathbf{E}_\alpha)_{\alpha\in I}$ is a family of spectra, then we can define their **product**

$$\mathbf{E} := \prod_{\alpha\in I} \mathbf{E}_\alpha$$

such that $E_i := \prod_{\alpha\in I} E_{\alpha,i}$. The structure maps of the product specrum are given by

$$E_i \cong \prod_{\alpha\in I} E_{\alpha,i} \stackrel{\simeq}{\to} \prod_{\alpha\in I} \Omega E_{\alpha,i+1} \cong \Omega \prod_{\alpha\in I} E_{\alpha,i+1} = \Omega E_{i+1} \ .$$

Here we use that Ω is a right-adjoint and therefore preserves products. For every $k \in \mathbb{Z}$ we have an isomorphism

$$H^k(X;\mathbf{E}) \cong \prod_{\alpha\in I} H^k(X;\mathbf{E}_\alpha) \ .$$

Example 4.10. If $\mathbf{E}$ is a spectrum and $k \in \mathbb{Z}$, then we can define the shift $\Sigma^k\mathbf{E}$ by $(\Sigma^k\mathbf{E})_i := E_{i+k}$. The structure maps of the shift are given by

$$(\Sigma^k\mathbf{E})_i = E_{i+k} \stackrel{\sim}{\to} \Omega E_{i+k+1} = \Omega(\Sigma^k\mathbf{E})_{i+1} \ .$$

We have a natural isomorphsm

$$H^i(X;\Sigma^k\mathbf{E}) \cong H^{i+k}(X;\mathbf{E}) \ .$$

Example 4.11. At the moment we shall accept that for every abelian group A there exists a spectrum $\mathbf{H}A$ with

$$H^k(S^0_*;\mathbf{H}A) \cong \left\{ \begin{array}{cc} A & k=0 \\ 0 & k \neq 0 \end{array} \right. \ .$$

This spectrum is called the **Eilenberg-MacLane spectrum of** A. In order to construct such a spectrum we must construct for every $n \in \mathbb{N}$ a so-called **Eilenberg-MacLane space** $K(A,n)$ and homotopy equivalences

$$K(A,n) \to \Omega K(A,n+1) \ .$$

The Eilenberg-MacLane space $K(A,n)$ is characterized by the property that $\pi_n(K(A,n)) \cong A$ is its only non-trivial homotopy group. If we had $K(A,n)$, then we could simply define $K(A,m) := \Omega^{n-m}K(A,n)$ for $m \leq n$. Unfortunately, there is no largest integer to start with.

A construction of the Eilenberg-MacLane spectrum $H\mathbf{A}$ will be given in Corollary 7.72.

More generally, if A is a $\mathbb{Z}$-graded group, then we can consider

$$\mathbf{H}A := \prod_{k\in\mathbb{Z}} \Sigma^k HA_k \ .$$

This is the Eilenberg-MacLane spectrum with coefficients in the $\mathbb{Z}$-graded group A.

The cohomology theory

$$X \mapsto H^*(X;A) := H^*(X;\mathbf{H}A)$$

is called the **ordinary cohomology** of X with coefficients in A.

□

In the following we discuss two features of the cohomology of unpointed spaces, namely excision and the Mayer-Vietoris sequence. For excision we consider a sequence of subspaces

$$U \subseteq A \subseteq X \ .$$

Lemma 4.12. *If we assume that $A \subset X$ is a NDR (see Definition 3.7) and that $U \subset A$ is open, then the natural map induces the **excision isomorphism***

$$H^*(X, A; \mathbf{E}) \cong H^*(X \setminus U, A \setminus U; \mathbf{E}) \ .$$

Proof. Note that $A \setminus U \to X \setminus U$ is again a NDR and the natural bijection

$$(X \setminus U)/(A \setminus U) \to X/A$$

is a homeomorphism.

Problem 4.13. *Check this assertion.*

Using Lemma 3.8 twice we get the isomorphism

$$H^*(X, A; \mathbf{E}) \cong H^*(X/A; \mathbf{E}) \cong H^*((X \setminus U)/(A \setminus U); \mathbf{E}) \cong H^*(X \setminus U, A \setminus U; \mathbf{E}) \ .$$

□

We now consider a decomposed space $X = U \cup V$. Let $i : U \to X$ and $j : V \to X$, $a : U \cap V \to U$ and $b : U \cap V \to V$ denote the inclusions.

Lemma 4.14. *If $V \subseteq X$ is a NDR and U is closed, then we have a long exact **Mayer-Vietoris sequence***

$$\cdots \to H^{k-1}(U \cap V) \stackrel{\delta}{\to} H^k(X) \stackrel{i^* \oplus j^*}{\to} H^k(U) \oplus H^k(V) \stackrel{a^* - b^*}{\to} H^{k-1}(U \cap V) \to \dots$$

Proof. We combine the long exact sequences of the pairs $(U, U \cap V)$ and (X, V) and excision for $(X \setminus U) \subseteq V \subseteq X$ by Lemma 4.12. We get the commuting diagram

$$\begin{tikzcd}
H^{k-1}(U\cap V) \arrow[r] \arrow[drr, dotted, "\delta"] & H^k(U,U\cap V) \arrow[r] & H^k(U) \arrow[r, "a^*"] & H^k(U\cap V) \\
H^{k-1}(V) \arrow[u] \arrow[r] & H^k(X,V) \arrow[u, "\cong"] \arrow[r] & H^k(X) \arrow[u, "i^*"] \arrow[r, "j^*"] & H^k(V) \arrow[u, "b^*"]
\end{tikzcd}$$

which defines δ as the obvious composition. Form this we derive exactness of the sequence

$$H^{k-1}(U \cap V) \stackrel{\delta}{\to} H^k(X) \stackrel{i^* \oplus j^*}{\to} H^k(U) \oplus H^k(V) \stackrel{a^* - b^*}{\to} H^{k-1}(U \cap V)$$

by a diagram chase.

Problem 4.15. *Work out the details of this argument.*

□

Problem 4.16. *Let X be a pointed space. Then we have an isomorphism*

$$\Sigma X \cong C(X) \cup_X C(X) \ , \quad [t,x] \mapsto \begin{cases} [2t,x] & t \le 1/2 \\ [2-2t,x] & t \ge 1/2 \end{cases} \ .$$

Find the precise relation between the suspension isomorphism and the boundary operator in the Mayer-Vietoris sequence associated to this decomposition

Problem 4.17. *Calculate the cohomology $H^*(\Sigma_{g,k};\mathbb{Z})$ for an oriented surface $\Sigma_{g,k}$ of genus g with k boundary components. Use the presentation of $\Sigma_{g,k}$ by glueing g cylinders over S^1 to a two-sphere with $2g+k$ holes and the Mayer-Vietoris sequence.*

4.2 CW-complexes and the AHSS

Assume that we have a filtered space

$$X_0 \subseteq X_1 \subseteq X_2 \dots \subseteq X \ .$$

Then the cohomology has a natural decreasing filtration

$$\mathcal{F}^p H^*(X;\mathbf{E}) := \mathtt{ker}(H^*(X;\mathbf{E}) \to H^*(X_{p-1};\mathbf{E})) \ .$$

For every $j \in \mathbb{N}$ we get a long exact sequence

$$\dots \to H^i(X_{j+1},X_j;\mathbf{E}) \to H^i(X_{j+1};\mathbf{E}) \to H^i(X_j;\mathbf{E}) \to H^{i+1}(X_{j+1},X_j;\mathbf{E}) \to \dots$$

These long exact sequences fit together in an exact couple

$$\begin{tikzcd} \bigoplus_j H^*(X_j;\mathbf{E}) \arrow[rr] & & \bigoplus_j H^*(X_j;\mathbf{E}) \arrow[dl] \\ & \bigoplus_j H^*(X_{j+1},X_j;\mathbf{E}) \arrow[ul] & \end{tikzcd} \ .$$

We thus get a spectral sequence in the right half space $(p,q) \in \mathbb{N} \times \mathbb{Z}$ with

$$E_1^{p,q} = H^{p+q}(X_p,X_{p-1};\mathbf{E}) \ .$$

If the filtration is finite, then the associated spectral sequence converges to

$$E_\infty^{p,q} \cong \mathrm{Gr}^p H^{p+q}(X;\mathbf{E}) \ .$$

This spectral sequence is useful if one can calculate $E_1^{p,q}$. This is possible for *CW*-complexes.

Definition 4.18. *A relative CW-**complex** (X,X_0) is a topological space X with additional structure:*

1. An increasing filtration

$$X_0 \subseteq X_1 \subseteq X_2 \cdots \subseteq X$$

indexed by $\mathbb{N}$.

2. For every $j \in \mathbb{N}$ *a collection of attaching maps* $(\kappa_\alpha)_{\alpha \in I_{j+1}}$, $\kappa_\alpha : S^j \to X_j$.

3. A presentation of the inclusion $X_j \to X_{j+1}$*as a pushout*

$$\begin{tikzcd} \bigsqcup_{\alpha \in I_{j+1}} S^j \arrow[r, "\sqcup_\alpha \kappa_\alpha"] \arrow[d] & X_j \arrow[d] \\ \bigsqcup_{\alpha \in I_{j+1}} D^{j+1} \arrow[r, "\sqcup_\alpha e_\alpha"] & X_{j+1} \end{tikzcd} \quad . \tag{10}$$

4. A homeomorphism $X \cong \mathtt{colim}_j X_j$.

If X_0 *is a discrete space, then we call* X *a CW-complex.*

Let X be a CW-complex. Then we call X_j the **j-skeleton** of X. For $j \in \mathbb{N}$ and $\alpha \in I_{j+1}$ the map $\kappa_\alpha : S^j \to X_j$ is called the **attaching map** of the cell with index α, and $e_\alpha : D^{j+1} \to X_{j+1}$ is the **characteristic map** of the same cell. The spectral sequence associated to the filtration of a CW-complex by its skeleta is called the **Atiyah-Hirzebruch spectral sequence** (AHSS). If the filtration is finite, then it converges to $\mathrm{Gr}H^*(X; \mathbf{E})$.

If X is a CW-complex, then the inclusion $X_j \to X_{j+1}$ is a NDR, see Example 3.10. Consequently

$$H^*(X_{j+1}, X_j; \mathbf{E}) \cong H^*(X_{j+1}/X_j; \mathbf{E}) \ .$$

Moreover, the presentation (10) induces a homeomorphism

$$\bigvee_{\alpha \in I_{j+1}} S^{j+1}_* \stackrel{\cong}{\leftarrow} \bigvee_{\alpha \in I_{j+1}} D^{j+1}/S^j \stackrel{\cong}{\to} X_{j+1}/X_j \ .$$

We therefore get, using the wedge axiom and the suspension isomorphism,

$$H^*(X_{j+1}/X_j; \mathbf{E}) \cong H^*(\bigvee_{\alpha \in I_{j+1}} S^{j+1}_*; \mathbf{E}) \cong \prod_{\alpha \in I_{j+1}} H^*(S^{j+1}_*; \mathbf{E}) \cong \prod_{\alpha \in I_{j+1}} E^{*-j-1} \ .$$

Putting these isomorphism together we get the first page of the AHSS

$$E_1^{p,q} \cong H^{p+q}(X_p, X_{p-1}; \mathbf{E}) \cong \prod_{\alpha \in I_p} E^q \ .$$

Our next task is to describe the differential

$$d_1 : E_1^{p,q} \to E_1^{p+1,q} \ .$$

In view of the above isomorphism we must describe homomorphism

$$\prod_{\alpha \in I_p} E^q \to \prod_{\alpha \in I_{p+1}} E^q \ .$$

Let $\alpha \in I_p$ and $\beta \in I_{p+1}$. Then it suffices to give the composition

$$\phi_{\beta\alpha} : E^q \hookrightarrow \prod_{\alpha' \in I_p} E^q \to \prod_{\beta' \in I_{p+1}} E^q \to E^q \ ,$$

where the first map is the natural inclusion of the factor with index α and the last map is the projection to the factor with index β. The following exercise shows that the collections of maps $(\phi_{\beta\alpha})$ really defines a morphisms between the products.

Problem 4.19. *Check that for every $\beta \in I_{p+1}$ the set*

$$\{\alpha \in I_p \mid \phi_{\beta\alpha} \neq 0\}$$

is finite.

If we follow through the definitions $\phi_{\beta\alpha}$ is given by the composition

$$\begin{aligned} E^q \cong H^{p+q}(S^p_*; \mathbf{E}) \stackrel{\pi_\alpha^*}{\to} H^{p+q}(X_p/X_{p-1}; \mathbf{E}) \to H^{p+q}(X_p; \mathbf{E}) \to H^{p+q+1}(X_{p+1}/X_p; \mathbf{E}) \to \\ \stackrel{i_{\beta *}}{\to} H^{p+q+1}(S^{q+1}_*; \mathbf{E}) \cong E^q \ . \end{aligned}$$

Here

$$\pi_\alpha : \frac{X_p}{X_{p-1}} \to \frac{X_p}{X_{p-1} \sqcup \coprod_{\alpha' \in I_p \setminus \{\alpha\}} e_{\alpha'}(D^p)} \cong S^p_*$$

is the natural projection where the last homeomorphism is induced by e_α. Furthermore,

$$i_\beta : S^{p+1}_* \cong D^{p+1}/S^p \stackrel{e_\beta}{\to} X_{p+1}/X_p \ .$$

The following square commutes

$$\begin{tikzcd}
H^{p+q}(X_p; \mathbf{E}) \arrow[r] \arrow[dddd, equal] & H^{p+q+1}(X_{p+1}/X_p; \mathbf{E}) \arrow[d, "e_{\beta *}"] \\
& H^{p+q+1}(D^{p+1}/S^p; \mathbf{E}) \\
& H^{p+q+1}(S^{p+1}_*; \mathbf{E}) \arrow[u, "\cong"'] \arrow[d, "\cong"', "suspension"] \\
& \\
H^{p+q}(X_p; \mathbf{E}) \arrow[r, "\kappa_\alpha^*"] & H^{p+q}(S^p_*; \mathbf{E})
\end{tikzcd}$$

Problem 4.20. *Check this assertion.*

It follows that $\phi_{\beta\alpha}$ is induced by the composition

$$\pi_\alpha \circ \kappa_\beta : S^p_* \to S^p_* \ .$$

We first consider this map in ordinary cohomology with coefficients in $\mathbb{Z}$. Let $f : S^p_* \to S^p_*$ be a map. It induces a homomorphism

$$\mathbb{Z} \cong H^p(S^p;\mathbb{Z}) \to H^p(S^p;\mathbb{Z}) \cong \mathbb{Z} \ ,$$

and is hence given by multiplication by a uniquely determined number $\deg(f) \in \mathbb{Z}$.

Definition 4.21. *We call* $\deg(f) \in \mathbb{Z}$ *the* ***degree*** *of* f.

We conclude that in this case $\phi_{\beta\alpha} : E^p \to E^p$ is multiplication with the degree $\deg(\pi_\alpha \circ \kappa_\beta) \in \mathbb{Z}$

Remark 4.22. It is known that the degree induces a bijection

$$[S^p_*, S^p_*] \stackrel{\cong}{\to} \mathbb{Z} \ .$$

We have not yet developed the techniques to show this assertion, but see Theorem 6.16 below. For $p \geq 1$ the bijection is an isomorphism of rings when consider S^p_* as a co-H-space for $p \geq 1$ for the additive structure, and the composition of maps for the multiplication on $[S^p_*, S^p_*]$.

Remark 4.23. For applications we must be able to calculate the degree of a map. Let $f : S^p \to S^p$ be given. Up to homotopy we can assume that there exists a point $\xi \in S^p$ and a small embedded p-ball $\xi \in B \subseteq S^p$ such that f is smooth on the preimage $f^{-1}(B)$. We can further assume that ξ is regular. After shrinking the balls we can now assume that $f_{|f^{-1}(B)}$ is a covering. We can now deform f (keeping its restriction to $f^{-1}(1/2B)$ fixed) by expanding it on $B \setminus 1/2B$ to a map

$$S^p \to \bigvee_{f^{-1}(\xi)} S^p \stackrel{\vee_x f_x}{\to} S^p \ ,$$

where the restriction f_x of f to the summand with index x is a diffeomorphism. Here the first map is an iteration of the coproduct $\delta : S^p \to S^p \vee S^p$. We get

$$\deg(f) = \sum_{x \in f^{-1}(\xi)} \deg(f_x) \ .$$

The degree of a diffeomorphism must be a unit in $\mathbb{Z}$, i.e. $\deg(f_x) \in \{1,-1\}$. The degree of the identity is 1. One can now check that the degree of an orientation reversing map is -1. One can e.g. produce a homotopy from $(\mathtt{id} \vee -\mathtt{id}) \circ \delta : S^p \to S^p \vee S^p \to S^p$ to the constant map.

Problem 4.24. *Give the details.*

Proposition 4.25. *The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ of the first page of the AHSS is given by the collection multiplication operators $\phi_{\beta\alpha} = \deg(\pi_\alpha \circ \kappa_\beta)$ for $\beta \in I_{p+1}$ and $\alpha \in I_p$.*

Proof. We have checked this in the case of ordinary cohomology with coefficients in $\mathbb{Z}$. The general case would require more theory which we have not developed so far. □

Let us specialize the AHSS to the case $\mathbf{E} = \mathbf{H}A$ for an abelian group A. In this case the whole spectral sequence is concentrated in the zero line, i.e. we have $E_*^{p,q} = 0$ for $q \geq 1$. We define the cochain complex

$$C^p(X;A) := E_1^{p,0} \cong \prod_{\alpha \in I_p} A \ , \quad d := d_1 : C^p(X;A) \to C^{p+1}(X;A) \ . \tag{11}$$

This is the **cochain complex of the CW-complex X with coefficients in** A. Note that the notation is sloppy since the chain complex depends on the cell decomposition of X. If the filtration is finite (this condition is actually not necessary), then we get an isomorphism

$$H^p(X;A) \cong \mathrm{Gr}^p H^p(X,A) \cong E_\infty^{p,0} \cong E_2^{p,0} \cong H^p(C^*(X,A),d) \ .$$

Note that $H^*(X;A)$ does not depend on the CW-structure.

Remark 4.26. By Proposition 4.25 (the unproven part), in the general case, the second page of the Atiyah-Hirzebruch spectral sequence of a CW-complex can be expressed in terms of ordinary cohomology:

$$E_2^{p,q} \cong H^p(X;E^q) \ .$$

From the second page on it does not depend on the CW-structure. In general there is now space for higher differentials which are usually difficult to determine.

4.3 Calculations of cohomology

Example 4.27. We consider the complex projective space $\mathbb{CP}^n$. We have a filtration

$$* \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \cdots \subset \mathbb{CP}^{n-1} \subset \mathbb{CP}^n \ .$$

We identify $\mathbb{C}^n \cong \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ such that $z \in \mathbb{C}^n$ corresponds to the line $[1:z] \in \mathbb{CP}^n$. The projection map $\mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$, $z \mapsto [z]$, extends to the compactification of $\mathbb{C}^n \hookrightarrow D^{2n}$ with the sphere at infinity. Hence we get a presentation

$$\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup_{S^{2n-1}} D^{2n} \ .$$

This equips the complex projective space $\mathbb{CP}^n$ with a CW-structure with one cell in every even dimension between 0 and $2n$.

The cellular cochain complex $C^*(\mathbb{CP}^n;\mathbb{Z})$ is now given by

$$\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \ ,$$

where the last $\mathbb{Z}$ is in degree $2n$. We read off the cohomology:

$$H^k(\mathbb{CP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 2i \ , \quad i \in \{0,1,\dots,n\} \\ 0 & \qquad\qquad else \end{cases} \ .$$

For an arbitrary group A we get the same picture:

$$H^k(\mathbb{CP}^n;A) \cong \begin{cases} A & k = 2i \ , \quad i \in \{0,1,\dots,n\} \\ 0 & \qquad\qquad else \end{cases} \ .$$

Example 4.28. The real projective space $\mathbb{RP}^n$ has a similar filtration

$$* \subset \mathbb{RP}^1 \subset \cdots \subset \mathbb{RP}^{n-1} \subset \mathbb{RP}^n \ .$$

This gives a CW-structure with one cell in every dimension between 0 and n.

The cellular cochain complex $C^*(\mathbb{RP}^n;\mathbb{Z})$ is given by

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to \mathbb{Z} \ ,$$

where the last $\mathbb{Z}$ is in degree n. It is now important to determine the differential. To this end we must study the map

$$\phi : S^j \to \mathbb{RP}^j \to \mathbb{RP}^j/\mathbb{RP}^{j-1} \cong S^j \ .$$

It is given on the open dense subset $\{\xi_0 \neq 0\}$ by

$$S^j \ni \xi \mapsto [\xi] \to [[\xi]] \to \xi/\xi_0 \in \mathbb{R}^j \subset S^j \ .$$

The first map sends a point in the sphere to the line determined by this point. The second map identifies all points of the form $[0:z]$ to one point. On this subset it is a two-fold covering, orientation preserving on the sheet $\xi_0 > 0$ and orientation preserving (reversing) on the sheet $\xi_0 < 0$ of j is even (odd). From this we deduce

$$\deg(\phi) = \begin{cases} 2 & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

Hence, the cochain complex is more precisely

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z} \ ,$$

We now read off the cohomology.

$$H^k(\mathbb{RP}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k=2i\ , \quad i\in\{1,2,\dots,[n/2]\} \\ \mathbb{Z} & n \text{ odd and } k=n \\ 0 & else \end{cases} .$$

In is interesting to consider the cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. In this case multiplication by 2 is 0 and the complex is given by

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \dots \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}\ .$$

We conclude that

$$H^k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k=i\ , \quad i\in\{0,1,\dots,n\} \\ 0 & \text{else} \end{cases} .$$

5 Fibre sequences

5.1 The homotopy fibre

We consider based spaces (X,x) and (Y,y) and a morphism between based spaces $f: X\to Y$. The fibre of f is the subspace $f^{-1}(y)\subseteq X$ with the base point x.

We can write the fibre as a pull-back

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ * \arrow[r, "y"] & Y \end{tikzcd} .$$

The fibre has a similar problem as the quotient. It is not homotopy invariant. The following example should be compared with Example 3.1.

Example 5.1. The constant map $\mathtt{const}_1 : S^1\to S^1$ with image $1\in S^1$ is homotopic to a map $f: S^1\to S^1$ which is a two-fold covering near 1.

Problem 5.2. *Find such a map.*

The fibre of the first is S^1, while the fibre of the second is a two-point space. These two spaces are not homotopy equivalent.

The fibre $f^{-1}(y)$ is characterized by the following universal property. For every based space T to give a map $\phi : T \to f^{-1}(y)$ is equivalent to give a map $\tilde{\phi} : T \to X$ such that the composition $f \circ \tilde{\phi}$ is constant.

In analogy with Remark 3.6 we want to define a space, called the homotopy fibre $\mathrm{Fib}(f)$ of f, with the universal property that to give a map $\phi : T \to \mathrm{Fib}(f)$ is the same as to give a map $\tilde{\phi} : T \to X$ together with a homotopy $h : [0,1] \times T \to Y$ from the constant map to the composition $f \circ \tilde{\phi}$.

We let $[0,1]$ be a based space with base point 0 and consider the **path space**

$$PY := \mathtt{Map}([0,1], Y) \ .$$

We have an evaluation $e : PY \to Y$ given by $e(\gamma) := \gamma(1)$. The path space PY is contractible. A contraction is given by

$$(s, \gamma) \mapsto (t \mapsto \gamma((1-s)t)) \ .$$

To give a map $T \to P$ is, by the exponential law, equivalent to give a map $T \to Y$ together with a homotopy of this map to the constant map.

Definition 5.3. *We define the* ***homotopy fibre*** $\mathrm{Fib}(f)$ *of a map* $f : X \to Y$ *between pointed spaces to be the based space given by the pull-back*

$$\begin{tikzcd} \mathrm{Fib}(f) \arrow[r, "i"] \arrow[d, "p"] & X \arrow[d, "f"] \\ PY \arrow[r, "e"] & Y \end{tikzcd}$$

Thus a point in $\mathrm{Fib}(f)$ is a pair (γ, x) of a point $x' \in X$ and a path γ in Y from the base point y to $f(x)$.

Problem 5.4. *Show that to give a map* $T \to \mathrm{Fib}(f)$ *is the same as to give a map* $T \to X$ *together with a homotopy from the constant map to* $f \circ T$.

There is a natural map (the analog of (7))

$$p : f^{-1}(y) \to \mathrm{Fib}(f) \ , \quad x' \mapsto (\mathtt{const}_y, x') \ .$$

We can again ask under which conditions this map is a homotopy equivalence.

Recall that a map f has the homotopy lifting property for a space A of the lift $\tilde{h}$ in

$$\begin{tikzcd} A \arrow[r, "\tilde{h}_0"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ I \times A \arrow[r, "h"] \arrow[ur, dotted, "\tilde{h}"] & Y \end{tikzcd}$$

for every choice of $h, \tilde{h}_0$. We call f a **fibration** if it has the homotopy lifting property for every space A.

Proposition 5.5. *If f is a fibration, then the map $p : f^{-1}(y) \to \mathrm{Fib}(f)$ is a homotopy equivalence.*

Proof. We must construct a homotopy inverse q for p. To this end we consider

$$\begin{tikzcd} \mathrm{Fib}(f) \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ I \times \mathrm{Fib}(f) \arrow[r, "h"] \arrow[ur, dotted, "\tilde{h}"] & Y \end{tikzcd} ,$$

where h is gien by

$$h(s,(\gamma,x')) := \gamma(1-s) \ .$$

We define

$$q := \tilde{h}_{|\{1\}\times \mathrm{Fib}(f)} : \mathrm{Fib}(f) \to f^{-1}(y) \ .$$

Then

$$I \times X \ni (s,x') \mapsto \tilde{h}(s,(\mathtt{const}_y,x')) \in X$$

is a homotopy from $\mathtt{id}_X$ to $q \circ p$. Similarly,

$$(s,(\gamma,x')) \mapsto ((t \mapsto \gamma((1-s)t)), \tilde{h}(s,(\gamma,x')))$$

is a homotopy from $\mathtt{id}_{\mathrm{Fib}(f)}$ to $p \circ q$. □

Lemma 5.6. *We can consider the* Fib *as a functor*

$$\mathrm{Fib} : \mathbf{Top}^{\Delta^1} \to \mathbf{Top}^{\Delta^1} \ , \quad (f : X \to Y) \mapsto (i : \mathrm{Fib}(f) \to X) \ .$$

Proof. Exercise. □

Assume that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

commutes up to distinguished homotopy H, i.e. is a generalized morphism $(\phi,\psi,H) : f \to f'$.

Lemma 5.7. *A generalized morphism $(\phi,\psi,H) : f \to f'$ determines commutative diagram*

$$\begin{tikzcd} \mathrm{Fib}(f) \arrow[r, "i"] \arrow[d, "\mathrm{Fib}(\phi,\psi,H)"] & X \arrow[d, "\phi"] \\ \mathrm{Fib}(f') \arrow[r] & X' \end{tikzcd} . \qquad (12)$$

Proof. We define

$$\mathrm{Fib}(\phi,\psi,H)(\gamma,x') := (\mu,\psi(f(x))\ ,$$

where μ is the concatenation of paths

$$t \mapsto \phi(\gamma(t))\ , \quad t \mapsto H(t,x)\ .$$

□

Lemma 5.8. *Given two composeable generalized morphisms $(\phi,\psi,H),(\phi',\psi',H')$ we have*

$$\mathrm{Fib}(\phi\circ\phi',\psi\circ\psi',H\circ H') \sim \mathrm{Fib}(\phi,\psi,H)\circ\mathrm{Fib}(\phi',\psi',H')\ .$$

Proof. Exercise. □

Lemma 5.9. *We have a homeomorphism $I\times\mathrm{Fib}(f) \stackrel{\cong}{\to} \mathrm{Fib}(I\times f)$*

Proof. It is given by

$$(t,(\gamma,x')) \mapsto ((t\mapsto(t,\gamma(t)),(t,f(x')))\ .$$

Corollary 5.10. *The map $(\phi,\psi,H)\mapsto\mathrm{Fib}(\phi,\psi,H)$ preserves homotopies.*

The following corollary in connection with Proposition 5.5 explains why Fib is the homotopy invariant way to take the fibre of a map.

Corollary 5.11. *If (ϕ,ψ,H) is a homotopy equivalence, then so is* $\mathrm{Fib}(\phi,\psi,H)$*.*

5.2 The fibre sequence

If $f: X\to Y$ is a map between pointed spaces, then we call

$$\mathrm{Fib}(f)\stackrel{i}{\to} X \stackrel{f}{\to} Y$$

the associated **fibre sequence**.

Lemma 5.12. *For a pointed space Z the sequence of pointed sets*

$$[Z,\mathrm{Fib}(f)] \stackrel{i_*}{\to} [Z,X] \stackrel{f_*}{\to} [Z,Y]$$

is exact.

Proof. To give a map $u : Z \to \mathrm{Fib}(f)$ is equivalent to give a map $v : Z \to Y$ and homotopy from the constant map to $f \circ v$. If we start with u, then $v = i \circ u$ and therefore $f \circ i \circ u \sim \mathtt{const}$. Vice versa, given v such that $v \sim const$, then the choice of a homotopy defines a preimage u such that $i \circ u = v$. $\square$

The following is completely analogous to Section 3.4. We can iterate the fibre construction and get the long fibre sequence

$$\mathcal{F}(f) \quad : \quad \cdots \stackrel{f^{(3)}}{\to} \mathrm{Fib}(f^{(1)}) \stackrel{f^{(2)}}{\to} \mathrm{Fib}(f^{(0)}) \stackrel{f^{(1)}}{\to} Y \stackrel{f^{(0)}}{\to} X \ .$$

Then

$$[Z, \mathcal{F}(f)]$$

is a long exact sequence of sets.

We now discuss the first iteration in detail

$$\mathrm{Fib}(i) \stackrel{k}{\to} \mathrm{Fib}(f) \stackrel{i}{\to} X \stackrel{f}{\to} Y \ .$$

There is a natural map

$$q : \Omega Y \to \mathrm{Fib}(f) \ , \quad \gamma \mapsto (\gamma, x) \ .$$

An point in $\mathrm{Fib}(i)$ is a tuple $(\mu, (\gamma, x'))$ where μ is a path in X from the base point to x' and γ is a path in Y from the base point to $f(x')$. We define a canonical map

$$a : \Omega Y \to \mathrm{Fib}(i) \ , \quad \gamma \mapsto (\mathtt{const}, (\gamma, x)) \ .$$

Lemma 5.13. *We have a commutative diagram*

$$\begin{tikzcd} \Omega Y \arrow[r, "q"] \arrow[d, "a"] & \mathrm{Fib}(f) \arrow[d, equal] \\ \mathrm{Fib}(i) \arrow[r, "k"] & \mathrm{Fib}(f) \end{tikzcd}$$

and a is a homotopy equivalence.

Proof. We give the inverse equivalence

$$b : \mathrm{Fib}(i) \to \Omega Y \ , \quad (\mu, (\gamma, x')) \mapsto f \circ \mu^{-1} \sharp \gamma \ .$$

The composition $a \circ b$ is given by

$$(\mu, (\gamma, x')) \mapsto (\mathtt{const}, (f \circ \mu^{-1} \sharp \gamma, x)) \ .$$

Let μ_s be the path μ up to time s. The a homotopy from $a \circ b$ to the identity is given by

$$(s, (\mu, (\gamma, x'))) \mapsto (\mu_s, (f \circ \mu_s^{-1} \sharp \gamma, \mu(s)))$$

Similarly, $b \circ a$ is given by

$$\gamma \mapsto f \circ \texttt{const}\sharp\gamma\ .$$

This is obviously homotopic to the identity. ∎

We now verify the dual of Lemma 3.24.

We have a diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "?"] \arrow[dd, bend right=60, "a^{(1)}", "\simeq"'] & \Omega Y \arrow[d, equal] & & & \\
\mathrm{Fib}(q) \arrow[r] \arrow[d] & \Omega Y \arrow[r, "q"] \arrow[d, "a", "\simeq"'] & \mathrm{Fib}(f) \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathrm{Fib}(k) \arrow[r, "\ell"] & \mathrm{Fib}(i) \arrow[r, "k"] & \mathrm{Fib}(f) \arrow[r, "i"] & X \arrow[r] & Y
\end{tikzcd}$$

Lemma 5.14. *The map ? is homotopic to $-\Omega(f)$.*

Proof. Exercise. ∎

Corollary 5.15. *We have equivalences $\Omega\mathrm{Fib}(f) \simeq \mathrm{Fib}(-\Omega f)$.*

This is the analog of Corollary 3.25.

Corollary 5.16. *If $f : X \to Y$ is a map of pointed spaces, then the long fibre sequence is equivalent to*

$$\dots \to \Omega^2\mathrm{Fib}(f) \stackrel{\Omega^2(i)}{\to} \Omega^2 X \stackrel{\Omega^2(f)}{\to} \Omega^2 Y \stackrel{-\Omega(q)}{\to} \Omega\mathrm{Fib}(f) \stackrel{-\Omega(i)}{\to} \Omega X \stackrel{-\Omega(f)}{\to} \Omega Y \stackrel{q}{\to} \mathrm{Fib}(f) \stackrel{i}{\to} X \stackrel{f}{\to} Y$$

Corollary 5.17. *If Z is a pointed space, then we have a long exact sequence*

$$\begin{tikzcd}
 & & {} \arrow[dll] \\
{[Z, \Omega^2\mathrm{Fib}(f)]} \arrow[r] & {[Z, \Omega^2 X]} \arrow[r] & {[Z, \Omega^2 Y]} \arrow[dll] \\
{[Z, \Omega\mathrm{Fib}(f)]} \arrow[r] & {[Z, \Omega X]} \arrow[r] & {[Z, \Omega Y]} \arrow[dll] \\
{[Z, \mathrm{Fib}(f)]} \arrow[r] & {[Z, X]} \arrow[r] & {[Z, Y]}
\end{tikzcd} \quad . \tag{13}$$

The lowest line consists of sets, the first line of groups, and all higher lines of abelian groups.

Remark 5.18. There is a right action of the group $[Z, \Omega Y]$ on $[Z, \mathrm{Fib}(f)]$. It is induced by the map

$$\Omega Y \times \mathrm{Fib}(f) \to \mathrm{Fib}(f) \ , \quad (\gamma, (\gamma', x')) \mapsto (\gamma' \sharp \gamma, x') \ .$$

The action is simply transitive on the fibre of $[Z, \Omega Y] \to [Z, \Omega X]$ over the base point.

5.3 The long exact homotopy sequence

We consider a sequence of maps $F \xrightarrow{j} X \xrightarrow{f} Y$ such that $f \circ j$ is homotopic to the constant map. A choice of a homotopy provides a lift in the diagram

$$\begin{tikzcd} F \arrow[r, dotted, "c"] \arrow[d, "j"] & \mathrm{Fib}(f) \arrow[d, "i"] \\ X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \ .$$

The sequence is called a **fibre sequence** if c is a homotopy equivalence. It is called a **quasi-fibration** of c is a weak equivalence.

Example 5.19. If the map $f : X \to Y$ is a fibration and $F \to X$ is the inclusion of the fibre over the base point, then $F \to X \to Y$ is a fibration sequence.

We consider a pull-back diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} \ .$$

Lemma 5.20. *If f is a fibration, then f' is a fibration.*

Proof. Exercise. □

Proposition 5.21. *If $F \to X \to Y$ is a quasi-fibration, then we have a long exact sequence of homotopy groups/sets.*

$$\dots \to \pi_2(F) \to \pi_2(X) \to \pi_2(X) \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y) \ .$$

Proof. We apply (13) to $Z = S^0$. We can replace $\mathrm{Fib}(f)$ by F. Finally we use that $[S_0^*, \Omega^k A] \cong \pi_k(A)$. □

We should provide the example that a locally trivial fibre bundle is a fibration sequence.

Theorem 5.22. *If $\pi : W \to B$ is a locally trivial fibre bundle over a paracompact Hausdorff space, then it is a fibration.*

Proof. For the very technical proof we refer to [Spa81, Cor. 2.7.14] □

6 Homotopy groups

6.1 Calculation of homotopy groups

Example 6.1. We have

$$\pi_i(S^n) = 0 \ , \quad 1 \le i < n \ .$$

To see this consider a base-point preserving map $f : S^i \to S^n$. We can deform f to a smooth map. By Sard's theorem its set of regular values has full measure and is therefore not empty. If $\xi \in S^n$ is a regular value, then because of $i = \dim(S^i) < \dim(S^n)$ we have $f^{-1}(\xi) = \emptyset$. We indentify S^n with the compactification of $\mathbb{R}^n$ at ∞ such that $\infty = \xi$. Then we can deform f to a constant map inside $\mathbb{R}^n$.

On the other hand we have seen in Subsection 4.2 that $\pi_n(S^n) \ne 0$.

Lemma 6.2. *If $f : X \to Y$ is a covering of pointed spaces, then $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \ge 2$ and injective for $i = 1$.*

Proof. A covering is a locally trivial fibre bundle with a discrete fibre which we will denote by F. We have $\pi_i(F) = 0$ for all $i \ge 1$. It now follows from the long exact homotopy sequence that f_* is an isomorphism between homotopy groups in degree ≥ 2 and injective for $i = 1$. □

Example 6.3. For $n \in \mathbb{N}$, $n \ge 1$, we have a convering $\mathbb{R}^n \to T^n$. Since $\mathbb{R}^n$ is contractible we can conclude that $\pi_i(T^n) \cong 0$ for $i \ge 2$. We have already seen that $\pi_1(T^n) \cong \mathbb{Z}^n$. Hence T^n is a model for the Eilenberg-MacLane space $K(\mathbb{Z}^n, 1)$, see see Example 4.11.

More generally, if $E\Gamma$ is a contractible space on which a discrete group Γ acts freely and properly, then $B\Gamma := E\Gamma/\Gamma$ has the homotopy type of $K(\Gamma, 1)$, i.e.

$$\pi_i(B\Gamma) \cong \left\{ \begin{array}{ll} 0 & i \ne 1 \\ \Gamma & i = 1 \end{array} \right. \ .$$

Such a space $B\Gamma$ is also called a **classifying space** for Γ.

Let $f : W \to B$ be a locally trivial fibre bundle with fibre S^1. Since $\pi_i(S^1) = 0$ for $i \ge 2$ we have isomorphisms

$$f_* : \pi_i(W \to \pi_i(B) \ , \quad i \ge 3 \ ,$$

and an exact sequence

$$0 \to \pi_2(W) \to \pi_2(B) \to \mathbb{Z} \to \pi_1(W) \to \pi_1(B) \to 0 \ .$$

Example 6.4. We can apply this to the Hopf bundle $S^{2n+1} \to \mathbb{CP}^n$. We conclude that

$$\pi_i(\mathbb{CP}^n) \cong \pi_i(S^{2n+1}) \quad i \geq 3 \ .$$

In particular,

$$\pi_i(\mathbb{CP}^n) \cong 0 \ , \quad 3 \leq i \leq 2n \ .$$

We further note that

$$\pi_2(\mathbb{CP}^n) \cong \mathbb{Z} \ , \quad \pi_1(\mathbb{CP}^n) \cong 0 \ .$$

There are natural inclusions $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$. We define

$$\mathbb{CP}^\infty := \mathtt{colim}_{n\to\infty}\mathbb{CP}^n \ .$$

Then

$$\pi_i(\mathbb{CP}^n) \cong \begin{cases} 0 & i \neq 2 \\ \mathbb{Z} & i = 2 \end{cases}$$

Hence $\mathbb{CP}^\infty$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, see Example 4.11.

Let $f : X \to Y$ be a map between spaces. For a choice of a base point in X we can define

$$\pi_i(f) := \pi_{i-1}(\mathrm{Fib}(f)) \ , \quad i \in \mathbb{N}$$

and get a corresponding long exact sequence. In particular, if f is the inclusion of a subspace, then we write

$$\pi_i(Y, X) := \pi_i(f)$$

and call them the **relative homotopy groups** (or sets in the case $i = 1$). We therefore have a long exact sequence of the pair (X, Y)

$$\cdots \to \pi_2(Y) \to \pi_2(X, Y) \to \pi_1(X) \to \pi_1(Y) \to \pi_1(Y, X) \to \pi_0(X) \to \pi_0(Y) \ .$$

We say that the map f is q**-connected** if and $\pi_0(X) \to \pi_0(Y)$ is surjective and $\pi_i(f) = 0$ for all $1 \leq i \leq q$ for all choices of a base point in X.

Remark 6.5. The condition that f is q-connected is equivalent to the condition that

1. $\pi_i(X) \to \pi_i(Y)$ is bijective for $i < q$ and
2. $\pi_i(X) \to \pi_i(Y)$ is surjective for $i = q$.

More generally, if $f : A \to B$ is a map between $\mathbb{Z}$-graded abelian groups, then we call it n-connected, if it induces an isomorphism in the components of degree $< n$ and a surjection in degree n.

Remark 6.6. We now give an explicit description of relative homotopy classes.

To give a map $S^n \to \mathrm{Fib}(f)$ is equivalent to give based maps $a : S^n \to Y^I$ and $b : S^n \to X$ such that $\mathtt{ev}_1 \circ a = f \circ b$. The map a is the same data as a map $a : I \times S^n \to Y$ such that $a(0, \xi) = y$, i.e. a map $a : D^{n+1} \cong C(S^n) \to Y$. This map must further satisfy $a_{|S^n} = f \circ b$.

We conclude:

Corollary 6.7. *For $n \in \mathbb{N}$ the group (or set if $n = 0$) $\pi_{n+1}(f)$ is the group (set) of homotopy classes of pairs of maps (a, b) where $a : D^{n+1} \to Y$ and $b : S^n \to X$ are such that $a_{|S^n} = f$.*

In the special case where f is the inclusion of a subset we have

$$\pi_{n+1}(f) \cong [(D^{n+1}, S^n), (Y, X)] \ .$$

We can further describe the natural transformations

$$\pi_{n+1}(f) \to \pi_n(X) \ , \quad \pi_{n+1}(Y) \to \pi_{n+1}(f)$$

explicitly. The first sends the class of (a, b) to the class of b. The second sends the class of a map $h : S^{n+1} \to Y$ to the pair $(a, const)$, where $a : D^{n+1} \to D^{n+1}/S^n \cong S^{n+1} \xrightarrow{b} Y$.

Example 6.8. We let $D \subset S^2$ be the upper hemisphere and $n \in D$ be the northpole. We choose the southpole s as the base point. The inclusion of the lower hemisphere is a homotopy equivalence

$$(D^2, S^1) \xrightarrow{\simeq} (S^2 \setminus \{n\}, D \setminus \{n\}) \ .$$

The long exact sequence of the pair (D^2, S^1) gives (since $\pi_2(D^2, s) \cong 0$ and $\pi_1(D^2, s) \cong 0$)

$$\pi_2(D^2, S^1) \cong \pi_1(S^1_*) \cong \mathbb{Z} \ .$$

On the other hand we have a homotopy equivalence

$$(S^2, s) \to (S^2, D) \ .$$

We conclude that (for the same reasons as above)

$$\pi_3(S^2, s) \cong \pi_2(S^2, D) \ .$$

Combining these two calculations we get

$$\pi_3(S^2, s) \cong \mathbb{Z} \ .$$

We see that for a manifold M the group $\pi_i(M)$ can be non-trivial also for $i > \dim(M)$. Moreover, **excision for homotopy is wrong**:

$$\mathbb{Z} \cong \pi_3(S^2, D) \neq \pi_3(S^2 \setminus \{n\}, D \setminus \{n\}) \cong \pi_3(D^2, S^1) \cong \pi_2(S^1_*) \cong 0 \ .$$

6.2 The Blakers-Massey theorem and applications

Let $Y = Y_1 \cup Y_2$ be a decomposition of a space Y into a union of open subspaces with non-empty intersection $Y_0 := Y_1 \cap Y_2$.

Theorem 6.9 (Blakers-Massey theorem). *We assume that for $p, q \geq 1$*

1. *(Y_1, Y_0) is p-connected*
2. *(Y_2, Y_0) is q-connected.*

Then the exision map $(Y_2, Y_0) \to (Y, Y_1)$ is $p+q$-connected.

Proof. The proof will be given later. □

Remark 6.10. Note that the stament, that $(Y_2, Y_0) \to (Y, Y_1)$ is $p+q$-connected means, that $\pi_n(Y_2, Y_0) \to \pi_n(Y, Y_1)$ is

$$\begin{array}{rr} \text{surjective for} & n = p+q \\ \text{bijective for} & 1 \leq n \leq p+q-1 \end{array}$$

Remark 6.11. We consider the decomposition $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$ with the base point in the equator. In order to apply the Blakers-Massey theorem we replace these subsets by homotopy equivalent open neighbourhoods.

We consider the commuting square

$$\begin{tikzcd} \pi_i(S^n) \arrow[r, "E"] & \pi_{i+1}(S^{n+1}) \arrow[d, "\cong"] \\ \pi_{i+1}(D^{n+1}_-, S^n) \arrow[u, "\partial", "\cong"'] \arrow[r, "\iota"] & \pi_{i+1}(S^{n+1}, D^{n+1}_+) \end{tikzcd}$$

(defining E) and the following statements:

$N(n)$: S^n is $n-1$-connected
$E(n)$: ι is $2n-1$-connected.

Remark 6.12. Note that we have shown $N(n)$ in Example 6.1 for all n using methods of differential topology. We will reprove this here as a consequence of the Blakers-Massey theorem.

Proposition 6.13. *The statements $N(n)$ and $E(n)$ hold for every $n \in \mathbb{N}$.*

Proof. We prove the proposition by induction on n. We have $N(1) = \mathit{true}$ since S^1 is connected. We apply Blakers-Massey to

$$(Y, Y_1, Y_2, Y_0) = (S^{n+1}, D^{n+1}_+, D^{n+1}_-, S^n)$$

(see Remark 6.11) for $p = q = n$. The assumptions are verifed using $N(n)$ by the following Lemma:

Lemma 6.14. *For every $i \geq 0$ and $n \geq 0$ we have the isomorphisms*

$$\partial : \pi_{i+1}(D^{n+1}_-, S^n) \to \pi_i(S^n) \ , \quad \iota : \pi_i(S^{n+1}) \to \pi_i(S^{n+1}, D^{n+1}_\pm) \ .$$

Proof. The first follows from the long exact pair sequence and the fact that D^{n+1}_- is contractible. The second follows from the fact that the map of pairs $(S^{n+1}, *) \to (S^{n+1}, D^{n+1}_\pm)$ is a homotopy equivalence of pairs. □

Blakers-Massey gives $E(n)$. The surjectivity statement now also implies $N(n+1)$. □

The homomorphism E is given by suspension. In general, for pointed spaces X, Y we define

$$\Sigma : [X, Y] \to [\Sigma X, \Sigma Y] \ , \quad f \mapsto \mathtt{id}_{S^1} \wedge f \ .$$

Theorem 6.15 (Freudenthal's Suspension Theorem)**.** *The suspension map*

$$\Sigma : \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is $2n-1$-connected.

Proof. This is assertion $E(n)$.

We can now finally calculate $\pi_n(S^n)$ for $n \geq 1$.

Theorem 6.16. *For $n \geq 1$ we have an isomorphism*

$$\deg : \pi_n(S^n) \stackrel{\cong}{\to} \mathbb{Z} \ .$$

The inverse of this isomorphism is given by

$$\mathbb{Z} \cong \pi_1(S^1) \stackrel{\Sigma^{n-1}}{\to} \pi_n(S^n)$$

Proof. Follows from Theorem 6.15 □

Recall the discussion of the degree in Remark 4.23.

Example 6.17. If

$$f : S^n \to S^n \ , \quad g : S^m \to S^m$$

are maps between the pointed spheres, then we have the identity

$$\deg(f \wedge g) = \deg(f) \deg(g) \ .$$

Let $p = n + m$ and consider the composition

$$h : S^p \cong S^n \wedge S^m \stackrel{flip}{\to} S^m \wedge S^n \cong S^p \ .$$

Then we have

$$\deg(h) = (-1)^{mn} .$$

The verification of these facts is an exercise.

Example 6.18. In Example 6.8 we have seen that $\pi_3(S^2) \cong \mathbb{Z}$. By Freudenthal for $n = 2$ we get a sequence of maps for $p \geq 3$

$$\pi_3(S^2) \to \pi_4(S^3) \cong \pi_{p+1}(S^p)$$

where the first map is surjective. It is known that the image of the first map is $\mathbb{Z}/2\mathbb{Z}$. At the moment we can not show this fact.

Example 6.19. Recall that by the Banach fixed point theorem a contractive map $f : D^n \to D^n$ has a fixed point. The Brouwer fixed point theorem shows that one can omit the assumption that f is contractive.

Theorem 6.20 (Brouwer fixed point theorem)**.** *A continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. We argue by contradiction. If f has no fixed point then we can define a family of maps

$$g_t : S^{n-1} \to S^{n-1} , \quad x \mapsto \frac{x - tf(x)}{\|x - tf(x)\|}$$

parametrized by $t \in [0, 1]$. Note that $g_0 = \mathtt{id}_{S^{n-1}}$ and g_1 extends to D^n. It follows that

$$0 = \deg(g_1) = \deg(g_0) = 1 .$$

This is false. □

6.3 Homotopy of classical groups

For every n we have a locally trivial fibre bundle

$$U(n) \to U(n+1) \to S^{2n+1} .$$

We consider the long exact sequence in homotopy. We get

1. $\pi_i(U(n)) \to \pi_i(U(n+1))$ is an isomorphism for $i \leq 2n-1$.
2. We have a sequence

$$\pi_{2n+1}(U(n+1)) \to \mathbb{Z} = \pi_{2n+1}(S^{2n+1}) \to \pi_{2n}(U(n)) \to \pi_{2n}(U(n+1)) \to 0 .$$

In particular, the map $U(n) \to U(n+1)$ is $2n$-connected.

We have $U(1) \cong S^1$. The inclusion $U(1) \to U(n)$ induces an isomorphism

$$\mathbb{Z} \cong \pi_1(U(1)) \stackrel{\cong}{\to} \pi_1(U(n))$$

for all $n \in \mathbb{N}$. We have $U(2) \cong U(1) \times S^3$ as spaces.

Problem 6.21. *Show this assertion.*

Hence,

$$0 \cong \pi_2(U(2)) \cong \pi_2(U(n))$$

for all $n \in \mathbb{N}$ with $n \geq 2$. Moreover,

$$\mathbb{Z} \cong \pi_3(U(2)) \stackrel{\cong}{\to} \pi_3(U(n))$$

for all $n \in \mathbb{N}$ with $n \geq 2$.

For the groups $SO(n)$ we can use the sequences

$$SO(n) \to SO(n+1) \to S^n \ .$$

We leave it to the reader to work out the consequences.

6.4 Quotients

Let (X, A) be a pair of spaces and $f : A \to B$ be a map. We consider a push-out diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "J"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

defining the pair (Y, B).

Proposition 6.22. *If (X, A) is p-connected and f is q-connected, then*

$$\pi_i(X, A) \to \pi_i(Y, B)$$

is $p + q$-connected.

Proof. We consider the mapping cylinder of f given by

$$Z(f) := (I \times A) \cup_{(0,a) \sim f(a)} B \ .$$

Note that we reverse the direction of the cylinder and attach B in the unusual manner at the bottom. It comes with an inclusion $k : A \to Z(f)$, $k(a) := (1, a)$, which is a cofibration and a projection $p : Z(f) \to B$ which is a homotopy equivalence. The mapping cylinder $Z(f)$ fits into the diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "j"] & Z(f) \arrow[r, "p"] \arrow[d, "L"] & B \arrow[d, "J"] \\ X \arrow[r, "K"] & Z \arrow[r, "P"] & Y \end{tikzcd} .$$

Here the left square defines the space Z as a push-out. The right square is the a push-out, too. Note that

$$Z \cong X \cup_{a\sim(1,a)} Z(f) \ .$$

The map P is obtained from the universal property of the push-out.

The map L is a push-out of the cofibration j and therefore itself a cofibration. The map P is a push-out of a homotopy equivalence p along the cofibration L and therefore again a homotopy equivalence. It follows that (p, P) is a homotopy equivalence between the morphisms L and J. Hence it suffices to show that the map $\pi_i(X, A) \to \pi_i(Z, Z(f))$ induced by (k, K) is $p + q$-connected.

We write (k, K) as composition

$$(X, A) \to ((0, 1] \times A \cup_{(1,a)\sim a} X, (0, 1] \times A) \xrightarrow{\iota} (Z, Z(f)) \ ,$$

where the first map sends $x \to x$ and the second map ι is determined by

$$(t, a) \mapsto (t, a) \in Z(f) \ , \quad x \mapsto x \ .$$

The first map is a homotopy equivalence. We must show that ι is $p + q$-connected.

We consider the diagram

$$\begin{tikzcd} ((0, 1] \times A \cup_{(1,a)\sim a} X, (0, 1) \times A) \arrow[r, "\iota'"] \arrow[d] & (Z, [0, 1) \times A \cup_{(0,a)\sim f(a)} B) \arrow[d] \\ ((0, 1] \times A \cup_{(1,a)\sim a} X, (0, 1] \times A) \arrow[r, "\iota"] & (Z, Z(f)) \end{tikzcd}$$

The vertical maps are homotopy equivalences. It suffices to show that ι' is $p+q$-connected.

To this end we apply the Blakers-Massey Theorem 6.9 to

1. $Y := Z$
2. $Y_1 := [0, 1) \times A \cup_{(0,a)\sim f(a)} B$
3. $Y_2 := (0, 1] \times A \cup_{(1,a)\sim a} X$
4. $Y_0 := (0, 1) \times A$.

By homotopy invariance $\pi_*(X,A) \cong \pi_*(Y_2,Y_0)$ and $\pi_*(f) \cong \pi_*(Y_1,Y_0)$. $\square$

We consider a pair of spaces $A \subset X$ and the map $(X,A) \to (X/A,*)$. In the following theorem we assume that A is connected and that the inclusion $A \to X$ is a cofibration.

Theorem 6.23. *If for $p,q \geq 1$ the pair $(C(A),A)$ is p-connected and (X,A) is q-connected, then*

$$\pi_i(X,A) \to \pi_i(X/A)$$

is $p+q$-connected.

Proof. We apply Proposition 6.22 to the map

$$(X,A) \to (X \cup C(A), C(A))$$

and conclude that it is $p+q$-connected. Since $C(A) \to X \cup C(A)$ is a cofibration and $C(A)$ is contractible the projection

$$X \cup C(A) \to X \cup C(A)/C(A) \cong X/A$$

is a homotopy equivalence. $\square$

We now consider a well-pointed space $(X,*)$, i.e. the inclusion $* \to X$ is a cofibration.

Theorem 6.24 (generalized Freudenthal)**.** *If $(X,*)$ is n-connected, then the suspension map*

$$\Sigma : \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

in homotopy is $2n+1$-connected.

Proof. We have a push-out diagram

$$\begin{tikzcd}
(\{0\} \times X) \vee I \times \{*\} \arrow[r] \arrow[d] & X \arrow[d] \\
I \times X \arrow[r] & C(X)
\end{tikzcd}$$

Since the left vertical map is a cofibration we conclude that the right vertical map is a cofibration, too. The suspension map is the conmposition

$$\pi_j(X) \overset{\cong,\partial}{\leftarrow} \pi_{j+1}(C(X),X) \overset{p}{\to} \pi_{j+1}(\Sigma X) \ ,$$

where $p : (C(X),X) \to (\Sigma X,*)$ is the natural projection. By assumption, Since $(X,*)$ is n-connected the pair $(C(X),X)$ is n-connected. We now apply Theorem 6.23 and conclude that $\pi_{*+1}(C(X),X) \overset{p}{\to} \pi_{*+1}(\Sigma X)$ is $2n$-connected. This implies the assertion after degree-shift by one. $\square$

Lemma 6.25. *For every $k \geq 0$ the space $\Sigma^k X$ is $k-1$-connected.*

Proof. We argue by induction on k. The case $k = 0$ is trivial. Let us now assume for $k \geq 0$ that $\Sigma^k X$ is $k-1$-connected. Then $0 \cong \pi_{k-1}(\Sigma^k X) \to \pi_k(\Sigma^{k+1} X)$ is an isomorphism since $k - 1 \leq 2(k-1) + 1$. $\square$

Example 6.26. For a pointed space X and $n \in \mathbb{N}$ we have a sequence of maps

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{k+2}(\Sigma^2 X) \xrightarrow{\Sigma} \dots \ .$$

We define the **stable homotopy groups** of X by

$$\pi_n^s(X) := \mathtt{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X) \ .$$

Using Lemma 6.25 and the generalized Freudenthal theorem 6.24 we see that $\pi_{n+k}(\Sigma^k X) \to \pi_{n+k+1}(\Sigma^{k+1} X)$ is an isomorphism if $k \geq n+1$. We conclude that

$$\pi_{2n+1}(\Sigma^{n+1} X) \cong \pi_n^s(X) \ .$$

Hence the colimit defining the nth stable homotopy group stabilizes from $k = n+1$ on.

6.5 Proof of the Blakers-Massey theorem

We follow the presentation of tom Dieck [tD08, Ch. 6.9].

For $n \in \mathbb{N}$ let $I^n := [0,1]^n \subset \mathbb{R}^n$ denote the standard cube. A **good embedding** $\mathbb{R}^p \to \mathbb{R}^n$ is a composition

$$scaling \circ translation \circ i \ ,$$

where $i : \mathbb{R}^p \to \mathbb{R}$ is an embedding which maps the standard basis of $\mathbb{R}^p$ to a subset of the standard basis of $\mathbb{R}^n$. A **cube** W of dimension $\dim(W) = p$ in $\mathbb{R}^n$ is the image of $I^p \subset \mathbb{R}^p$ under a good embedding. The boundary ∂W is the image of the boundary ∂I^p. It is the union of (closed) **boundary faces** which are again cubes of dimension $< p$.

Finally we define for a p-dimensional cube W

$$\begin{aligned} K_q(W) &:= \text{image of } \{x \in I^p \mid \sharp(i \in \{1,\dots,p\} \mid x_i < 1/2) \geq q\} \\ G_q(W) &:= \text{image of } \{x \in I^p \mid \sharp(i \in \{1,\dots,p\} \mid x_i < 1/2) \geq q\} \end{aligned}$$

Let W be a cube.

Lemma 6.27. *We consider a map $f : W \to Y$ and a subset $A \subset Y$. We assume that for $p \leq \dim(W)$ we have*

$$f^{-1}(A) \cap W' \subset K_p(W')$$

for all faces of W' in ∂W. Then there exists a map $g : W \to Y$ such that $f \sim g$ (rel ∂W) and

$$g^{-1}(A) \subset K_p(W) \ .$$

A similar statement holds for $G_p(W)$ in place of $K_p(W)$.

Proof. Let $n = \dim(W)$ and assume that $W = I^n$. We consider the point $x = (\frac{1}{4}, \ldots, \frac{1}{4})$. A ray y starting at x meets $\partial(1/2I^n)$ in the point $P(y)$ and ∂I^n in the point $Q(y)$. The map g is defined be the following prescription:

1. It sends the segment $\overline{P(y)Q(y)}$ to $Q(y)$.
2. It sends the segment $\overline{xP(y)}$ affinely to $\overline{yQ(y)}$.

Then $h \sim \mathtt{id}_{I^n}$ $(rel\ \partial I^n)$. We define $g := f \circ h$. One verifies that g does the job. Assume that $z \in I^n$ and $g(z) \in A$. We must show that $z \in K_p(W)$. Note that $1/2I^n \subseteq K_p(W)$. If $z \notin 1/2I^n$, then $h(z) \in \partial I^n$ and therefore is contained in some boundary face W'. Consequently $f(h(z)) \in A$ and therefore by assumption $h(z) \in K_p(W')$. Let $(h(z)_1, \ldots, h(z)_{\dim(W')}) \in I^{\dim(W')}$ be the coordinates of W'. Then $\sharp(j \mid h(z)_j < 1/2) \geq p$. Note $h(z)_i = 1/4 + t(z_i - 1/4)$ for some $t > 1$. Hence $\sharp(i \mid z_i < 1/2) \geq p$. This implies $z \in K_p(W)$. $\square$

We assume now that $Y = Y_1 \cup Y_2$ is a decomposition of a space into open subsets with $Y_0 := Y_1 \cap Y_2$. Let $f : I^n \to Y$ be given. We can choose a subdivision of I^n into cubes such that either $f(W) \subset Y_1$ or $f(W) \subset Y_2$ for all cubes W of the subdivision.

Proposition 6.28. *We assume that for $p, q \geq 0$ the pair (Y_1, Y_0) is p-connected and (Y_2, Y_0) is q-connected. Then there exists a homotopy f_t from f to f_1 such that the following holds:*

1. *If $f(W) \subset Y_j$, then $f_t(Y_j) \subset Y_j$ for all $t \in [0, 1]$.*
2. *If $f(W) \subset Y_0$, then f_t is constant on W.*
3. *If $f(W) \subset Y_1$, then $f_1^{-1}(Y_1 \setminus Y_0) \cap W \subset K_{p+1}(W)$.*
4. *If $f(W) \subset Y_2$, then $f_1^{-1}(Y_2 \setminus Y_0) \cap W \subset G_{q+1}(W)$.*

Proof. Let C_k be the union of all cubes W of dimension $\dim(W) \leq k$. We perform inductively on k the following construction steps.

1. We assume that a homotopy f_t is constructed such that 1. to 2. hold true and 3. and 4. hold true for the restriction of the homotopy to C_k.
2. For every cube W of dimension $k+1$ we construct a homotopy of $(f_1)_{|W}$ relative to the boundary such that 3. and 4. holds.

3. Using the fact that the inclusion $C_{k+1} \to I^n$ has the homotopy extension property these homotopies together with the constant homotopy on C_k can be extended to a homotopy of f_1 on all of I^n respecting 1. and 2. As long as $k \le p$ we assume for the cubes with $f_1(W) \subseteq Y_1$ that $f_1(\partial W) \subseteq Y_0$.

4. We concatenate this homotopy with f_t and call the result again f_t.

5. We increase k by one and start again.

We can start the iteration at $k = -1$. Then $C_{-1} = \emptyset$ and f_t is the constant homotopy of f. The iteration finishes at $k = n$.

We now argue that we can perform the step 2. Let W be a cube of dimension k.

1. If $f_1(W) \subset Y_0$, then we use the constant homotopy.

2. If $f_1(W) \subset Y_1$, $f_1(W) \not\subset Y_0$, and $k \le p$, then since (Y_1, Y_0) is p-connected and $f_1(\partial W) \subset Y_0$ there exists a homotopy f_t^W of $(f_1)_{|W}$ which is constant on ∂W and such that $f_1^W(W) \subset Y_0$.

3. If $f_1(W) \subset Y_1$, $f_1(W) \not\subset Y_0$, and $k \ge p+1$, then we find a homotopy f_t^W of $(f_1)_{|W}$ by Lemma 6.27.

□

We consider the space $F(Y_1, Y, Y_2)$ defined by the pull-back

$$\begin{tikzcd} F(Y_1,Y,Y_2) \arrow[r] \arrow[d] & Y^I \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ Y_1 \times Y_2 \arrow[r] & Y \times Y \end{tikzcd} .$$

Proposition 6.29. *We assume that for $p, q \ge 0$ the pair (Y_1, Y_0) is p-connected and the pair (Y_2, Y_0) is q-connected. Then the natural inclusion*

$$F(Y_1, Y_1, Y_0) \to F(Y_1, Y, Y_2)$$

is $p + q - 1$-connected.

Proof. Let $n \le p + q - 1$ and a map

$$\phi : (I^n, \partial I^n) \to (F(Y_1, Y, Y_2), F(Y_1, Y_1, Y_0))$$

be given. We must show that ϕ can be deformed as a map of pairs to a map which sends I to $F(Y_1, Y_1, Y_0)$.

By adjunction, ϕ corresponds to a map $\Phi : I^n \times I \to Y$ with

1. $\Phi(x, 0) \in Y_1$ for $x \in I^n$.

2. $\Phi(x,1) \in Y_2$ for $x \in I^n$.

3. $\Phi(y,t) \in Y_1$ for $y \in \partial I^n$ and $t \in I$.

Let us call maps with this property admissible. We must show that we can deform Φ inside admissible maps to a map which takes values in Y_1. We first apply Propostion 6.28 and get a map Ψ. Let $\pi : I^n \times I \to I^n$ be the projection.

We claim that
$$\pi(\Psi^{-1}(Y \setminus Y_1)) \cap \pi(\Psi^{-1}(Y \setminus Y_2)) = \emptyset \ .$$
Let y by a point in this intersection. Assume that $y = \pi(z)$ for $z \in \Psi^{-1}(Y \setminus Y_2) \cap W$ for some cube of the subdivision. Then $z \in K_{p+1}(W)$ and $y = \pi(z)$ has at least p large coordinates. Similarly, if $y = \pi(z)'$ for $z' \in \Psi^{-1}(Y \setminus Y_1) \cap W'$ we conclude that y has at least q small coordinates. If $n < p + q$, then such a point can not exist. This shows the claim.

We have
$$\pi(\Psi^{-1}(Y \setminus Y_1)) \cap \partial I^n = \emptyset$$
since $\Psi(\partial I^n \times I) \subseteq Y_1$. We choose a function $\tau : I^n \to I$ such that

1. $\tau \equiv 0$ on $\pi(\Psi^{-1}(Y \setminus Y_1))$
2. $\tau \equiv 1$ on $\partial I^n \cup \pi(\Psi^{-1}(Y \setminus Y_2))$.

Then we define the homotopy
$$(s,(x,t)) \mapsto \Psi(x,(1-s)t + st\tau(x)) \ .$$
It stays inside admissible maps and takes values in Y_1 for $s = 1$. $\square$

We can now finish the proof of the Blaker-Massey theorem 6.9. We consider the path fibration sequence
$$F(*,Y,Y_2) \to F(Y,Y,Y_2) \stackrel{\mathtt{ev}_0}{\to} Y \ .$$
Since the pull-back along $Y_1 \to Y$ of this fibration sequence is again a fibration sequence (Lemma 5.20) we get the map of fibration sequences

$$\begin{tikzcd}
F(*,Y_1,Y_0) \arrow[r, "\beta"] \arrow[d] & F(*,Y,Y_2) \arrow[d] \\
F(Y_1,Y_1,Y_0) \arrow[r, "\alpha"] \arrow[d, "\mathtt{ev}_0"] & F(Y_1,Y,Y_2) \arrow[d, "\mathtt{ev}_0"] \\
Y_1 \arrow[r, equal] & Y_1
\end{tikzcd} \ .$$

We have seen in Proposition 6.29 that α is $p+q-1$-connected. It follows by a diagram chase in the map of associated long exact homotopy sequences that β is $p+q-1$-connected, too. Finally we note that

$$\pi_n(F(*,Y_1,Y_0)) \cong \pi_{n+1}(Y_1,Y_0)\ , \quad \pi_n(F(*,Y,Y_2)) \cong \pi_{n+1}(Y,Y_2)\ .$$

□

7 Diverse Constructions

7.1 The $\Omega^\infty\Sigma^\infty$-construction

In Section 4.1 we have seen that a spectrum gives rise to a cohomology theory. In this section we describe the construction of suspension spectra.

A **prespectrum E** is a pair $((E_n)_{n\in\mathbb{N}}, (\sigma_n)_{n\in\mathbb{N}})$ consisting of a sequence of pointed spaces $(E_n)_{n\in\mathbb{N}}$ and maps $\sigma_n : \Sigma E_n \to E_{n+1}$ for all $n \in \mathbb{N}$. In contrast to spectra we do not require that the adjoint maps $E_n \to \Omega E_{n+1}$ are homotopy equivalences. Furthermore we only consider sequences indexed by $\mathbb{N}$ instead of $\mathbb{Z}$.

Example 7.1. Given a pointed space X we can construct the **suspension prespectrum** $\Sigma^\infty X$ simply by setting $(\Sigma^\infty X)_n := \Sigma^n X$ and $\sigma_n := \mathtt{id}_{\Sigma^{n+1}X}$.

We now describe a construction which associates to a prespectrum $\mathbf{E}$ a spectrum $R(\mathbf{E})$. We use the unit $u : \mathtt{id} \to \Omega\Sigma$ of the adjunction (Σ, Ω). For a pointed space X the map $u : X \to \Omega\Sigma X$ is the morphism corresponding to $\mathtt{id}_{\Sigma X}$ under the bijection

$$\mathtt{Hom}(\Sigma X, \Sigma X) \cong \mathtt{Hom}(X, \Omega\Sigma X)\ .$$

The data of a prespectrum provide maps

$$\Omega^k E_{n+k} \xrightarrow{u} \Omega^k \Omega\Sigma E_{n+k} \xrightarrow{\sigma_{n+k}} \Omega^{k+1} E_{n+k+1}$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ satisfying $n \geq -k$. We define

$$R(\mathbf{E})_n := \mathtt{colim}_{k\in\mathbb{N}, k\geq -n}\, \Omega^k E_{n+k}\ . \tag{14}$$

Since S^1 is compact the loop space functor Ω commutes with filtered colimits. For every $n \in \mathbb{Z}$ we get a sequence of homeomorphisms

$$\begin{aligned} \Omega R(\mathbf{E})_{n+1} &\cong \Omega\mathtt{colim}_{k\in\mathbb{N}, k\geq -n-1} \Omega^k E_{n+1+k} \\ &\cong \mathtt{colim}_{k\in\mathbb{N}, k\geq -n-1} \Omega^{k+1} E_{n+1+k} \\ &\cong \mathtt{colim}_{k\in\mathbb{N}, k\geq -n} \Omega^k E_{n+k} \\ &\cong R(\mathbf{E})_n\ . \end{aligned}$$

The adjoint of this homeomorphism is the structure map

$$\Sigma R(\mathbf{E})_{n-1} \to R(\mathbf{E})_n$$

of the spectrum $R(\mathbf{E})$.

Definition 7.2. *We call $R(\mathbf{E})$ the associated spectrum of $\mathbf{E}$. For a pointed space X the spectrum $R(\Sigma^\infty X)$ is called the suspension spectrum of X.*

Example 7.3. The **sphere spectrum $\mathbf{S}$** is defined as the suspension spectrum $\mathbf{S} := R(\Sigma^\infty S^0)$. The cohomology theory represented by $\mathbf{S}$ is the **stable cohomotopy** theory (or framed bordism theory). It is in a certain sense the simplest spectrum to define, but a very complicated cohomology theory. Its coefficients

$$\pi_n^s(\mathbf{S}) := H^0(S^n; \mathbf{S})$$

are called **the stable homotopy groups of the sphere**. They are calculated in small dimensions, say $n \leq 55$.

Problem 7.4. *Let X be compact and*

$$Y_0 \to Y_1 \to Y_2 \to \dots$$

a sequence of inclusions indexed by $\mathbb{N}$. Show that the natural map

$$\mathtt{colim}_{n\in\mathbb{N}}[X, Y_n] \to [X, \mathtt{colim}_{n\in\mathbb{N}} Y_n] \tag{15}$$

is a bijection.

If $\mathbf{E}$ is a prespectrum, then for every $n, k \in \mathbb{N}$ we have natural maps

$$[X, E_n] \xrightarrow{\Sigma} [\Sigma X, \Sigma E_n] \xrightarrow{\sigma_n} [\Sigma X, E_{n+1}] \ .$$

We can express the cohomology of a space X with coefficients in a the associated spectrum of a prespectrum $\mathbf{E}$ in terms of the prespectrum as follows:

Lemma 7.5. *If $\mathbf{E}$ is a prespectrum such that its structure maps are inclusions, then for a compact pointed space X we have*

$$H^k(X; R(\mathbf{E})) \cong \mathtt{colim}_{k\in\mathbb{N}, k\geq -n}[\Sigma^k X, E_{n+k}] \ . \tag{16}$$

Proof. Note that the unit $u : Y \to \Omega\Sigma Y$ is always an inclusion of subspaces. The assumption on $\mathbf{E}$ implies that the connecting maps $\Omega^k E_{n+k} \to \Omega^{k+1} E_{n+k+1}$ in the colimit defining $R(\mathbf{E})_n$ are inclusions of subspaces. We calculate

$$\begin{aligned} H^k(X; R(\mathbf{E})) &= [X, R(\mathbf{E})_n] \\ &\cong [X, \mathtt{colim}_{k\in\mathbb{N}, k\geq -n} \Omega^k E_{n+k}] \\ &\overset{!}{\cong} \mathtt{colim}_{k\in\mathbb{N}, k\geq -n}[X, \Omega^k E_{n+k}] \\ &\cong \mathtt{colim}_{k\in\mathbb{N}, k\geq -n}[\Sigma^k X, E_{n+k}] \end{aligned}$$

We use the compactness of X at the marked isomorphism (see Problem 7.4). $\square$

Remark 7.6. In the remark we comment on the condition that the structure maps of $\mathbf{E}$ are inclusions. In general we want that the cohomology of a spectrum associated to a prespectrum $\mathbf{E}$ is given by the right-hand side of (16). If the structure maps of $\mathbf{E}$ are not inclusions, then the spectrum $R(\mathbf{E})$ might be the wrong choice. In this case one would first modify the prespectrum such that the structure map become inclusions without changeing the right-hand side of (16) and then apply the construction of an associated spectrum.

Example 7.7. For a space X we have

$$H^0(S^n; R(\Sigma^\infty X)) \cong \mathtt{colim}_{k\in\mathbb{N}}[S^{n+k}, \Sigma^k X] \ .$$

By Example 6.26 we know that

$$H^0(S^n; R(\Sigma^\infty X)) \cong [S^{2n+1}, \Sigma^{n+1} X] \ .$$

Example 7.8. Assume that we are given a pointed space X with a map $\beta : \Sigma^\ell X \to X$ for some $\ell \in \mathbb{N}$, $\ell > 0$. Then using the unit u k times we get maps

$$\Omega^n X \to \Omega^{n+\ell}\Sigma^\ell X \to \Omega^{n+\ell} X \ .$$

We can define a spectrum $\Sigma^\infty X[\beta^{-1}]$ whose n space is

$$\Sigma^\infty X[\beta^{-1}]_n := \mathtt{colim}_k \Omega^{n+k\ell} X \ .$$

7.2 Simplicial sets, singular complex and geometric realization

Simplicial sets provide a combinatoral way to describe topological spaces. By $[n]$ we denote the poset $\{0, 1, \ldots, n\}$ with the natural order. We consider the category Δ whose objects are the posets $[n]$ for $n \in \mathbb{N}$ and whose morphisms are order preserving maps.

The following maps play a particular role:

1. $\partial_i : [n] \to [n+1]$ for $i = 0, \ldots, n+1$ is the injective map whose image does not contain i. It is called the ith **boundary face map**.
2. $\sigma_i : [n+1] \to [n]$ for $i = 0, n+1$ is the surjective map which sends i and $i+1$ to i. It is called the ith **degeneration**.

Definition 7.9. *For a category* $\mathbf{C}$ *we define the category of* ***simplicial objects*** $\mathbf{sC}$ *in* $\mathbf{C}$ *to be the functor category*

$$\mathbf{sC} := \mathbf{Fun}(\Delta^{op}, C) \ .$$

Similarly, the category

$$\mathbf{cC} := \mathbf{Fun}(\Delta, C)$$

is the category of cosimplicial objects in $\mathbf{C}$.

So one can consider **sC** as the category of presheaves with values in **C** on Δ.

Remark 7.10. One can check that every morphism in Δ can be written as a composition of a collection of boundary face maps followed collection of degenerations. Hence in order to present a simplicial or cosimplicial object in **C** it suffices to provide a sequence of objects $(X[n])_{n\in\mathbb{N}}$ and to describe the action of the boundary face maps and of the degenerations.

Example 7.11. Assume that the category **C** has fibre products and $X \to Y$ is a morphism in **C**. Then we can form the simplicial object $E(X \to Y)$ given by

$$E(X \to Y)[n] := \underbrace{X \times_Y \cdots \times_Y X}_{n+1} \ .$$

If $f : [n] \to [m]$ is a morphism in Δ, then the induced map

$$E(X \to Y)[m] \to E(X \to Y)[n]$$

is given by

$$(x_0, \ldots, x_m) \to (x_{f(0)}, \ldots, x_{f(m)}) \ .$$

We will in particular deal with the categories of simplicial sets **sSet** and simplicial topological spaces **sTop**.

Example 7.12. We define the cosimplicial topological space

$$\Delta^{-}_{top} : \Delta \to \mathbf{Top}$$

as follows:

1. Δ^n_{top} is the space of probablity measures on the underlying set $[n]$. Let δ_i be the point measure on $i \in [n]$. Then we can write every point in Δ^n_{top} in the form $\sum_{i=0}^n x_i\delta_i$. In this way we have coordinates $x = (x_0, \ldots, x_n)$ on Δ^n_{top}. Via these coordinates Δ^n_{top} can be considered as the subset of $\mathbb{R}^{n+1}$ determined by the conditions that $x_i \geq 0$ for all $i \in \{0, \ldots, n\}$ and $\sum_{i=0}^n x_i = 1$.
2. The functor Δ^{-}_{top} sends a map $f : [n] \to [m]$ to the push-forward map $\Delta_{top}(f) := f_*$ of measures. In coordinates the push-forward is given by $(f_*(x))_j = \sum_{i\in f^{-1}(j)} x_i$.

Remark 7.13. *The point of the description in terms of measures is to formulate the functoriality in a condensed way.*

The space Δ^n_{top} is called the n-dimensional **topological simplex**. The topological simplex Δ^n_{top} has the structure of an n-dimensional manifold with corners. The map

$$\Delta_{top}(\partial_i) : \Delta^n_{top} \to \Delta^{n+1}_{top}$$

is the inclusion of the ith boundary face. In coordinates it is given by

$$x \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n) \ .$$

The degeneration

$$\Delta_{top}(\sigma_i) : \Delta^{n+1}_{top} \to \Delta^n_{top}$$

is a linear projection to the ith boundary face $\{x_i = 0\}$ from the i'th corner $\{x_i = 1\}$ opposite to that boundary. In coordinates,

$$x \mapsto (x_0, \dots, x_i + x_{i+1}, \dots, x_{n+1}) \ .$$

The topological simplex Δ^n_{top} is homemeomorphic to D^n.

Problem 7.14. *Write out an explicit formula for the homeomorphism.*

Restricting the homemorphism to the boundary $\partial\Delta^n_{top}$ we get a decomposition of S^{n-1} into the images of the boundary faces. This is a triangulation of S^{n-1}, a special form of a cell decomposition.

We now consider constructions of simpicial sets.

1. The Yoneda embedding $\Delta^- : \Delta \to \mathbf{sSet}$ is given by $[n] \to \Delta^n := \mathtt{Hom}_\Delta(\dots, [n])$.
2. The Yoneda embeddig for **Top** is a functor

   $$\mathbf{Top} \to \mathbf{Fun}(\mathbf{Top}^{op}, \mathbf{Set}) \ , \quad X \mapsto \mathtt{Hom}_{\mathbf{Top}}(-, X) \ .$$

   The restriction of this functor along $\Delta^-_{top} : \Delta \to \mathbf{Top}$ yields a functor

   $$\mathtt{sing} : \mathbf{Top} \to \mathbf{sSet} \ .$$

   It associates to a topological space the simplicial set

   $$\mathtt{sing}(X)(-) := \mathtt{Hom}_{\mathbf{Top}}(\Delta^-_{top}, X) \ .$$

   Since $\Delta^-_{top} : \Delta \to \mathbf{Top}$ is covariant, this functor is indeed contravariant from $\Delta \to$ **Set**. The simplicial set $\mathtt{sing}(X)$ is called the **singular complex** of X.

3. We have a Yoneda embedding embedding

$$\mathbf{Cat} \to \mathbf{Fun}(\mathbf{Cat}, \mathbf{Set}) \ , \quad \mathbf{C} \mapsto \mathrm{Ob}\mathbf{Fun}(-, \mathbf{C}) \ ,$$

where Ob indicates that we take the set of objects of the functor category. Considering posets as categories we get a functor

$$\Delta \to \mathbf{Cat} \ , \quad [n] \mapsto [n] \ .$$

Restricting the Yoneda embedding along this functor we get the nerve functor

$$\mathtt{N} : \mathbf{Cat} \to \mathbf{sSet} \ .$$

We have

$$\mathtt{N}(C)[n] = \mathrm{Ob}\mathbf{Fun}([n], \mathbf{C}) \ .$$

Problem 7.15. *Describe the simplicial set* $\mathtt{N}(\mathbf{C})$ *explicitly.*

4. If G is a group, then we can form the category $\mathbb{B}G$ with one object and morphism set G, and whose composition is given by the group multiplication. Its nerve is the simplicial set

$$\mathbf{B}G := \mathtt{N}(\mathbb{B}G) \ .$$

5. We can consider G as a set and form the simplicial set $\mathbf{E}G := E(G \to *)$. The group G acts on this simplicial set. The quotient can be identified with BG.

Problem 7.16. *Give an explicit description of the sets* $\mathbf{B}G$ *and* $\mathbf{E}G$ *and show that* $\mathbf{E}G/G \cong \mathbf{B}G$ *and of the action of the boundary and face maps.*

Simplicial sets are used as combinatorial models of spaces. To this end we define a functor

$$|-| : \mathbf{sSet} \to \mathbf{Top}$$

as the **left Kan-extension** of $\Delta_{top} : \Delta \to \mathbf{Top}$ along the Yoneda embedding $\Delta^- : \Delta \to \mathbf{sSet}$ discussed above:

$$\begin{tikzcd} \Delta \arrow[rr, "\Delta_{top}"] \arrow[dr, "\Delta^-"'] & & \mathbf{Top} \\ & \mathbf{sSet} \arrow[ur, dotted, "|-|"'] & \end{tikzcd} \ .$$

It is called the **geometric realization** functor.

Lemma 7.17. *We have an adjunction*

$$|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \mathtt{sing} \ .$$

Proof. This is a formal consequence of the definition of $\mathtt{sing}$. Indeed, using the objectwise formula

$$|X| \cong \mathtt{colim}_{(\Delta \to X)} \Delta^-_{top}$$

for the left Kan extension and the general facts

$$X \cong \mathtt{colim}_{\Delta^- \to X} \Delta^- \ , \quad X \cong \mathtt{Hom}_{\mathbf{sSet}}(\Delta^-, X)$$

for $X \in \mathbf{sSet}$ (true for every category of presheaves of sets) we get for every topological space Y the following equivalence:

$$\begin{aligned}
\mathtt{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \mathtt{Hom}_{\mathbf{Top}}(\mathtt{colim}_{\Delta^- \to X} \Delta^-_{top}, Y) \\
&\cong \mathtt{lim}_{\Delta^- \to X} \mathtt{Hom}_{\mathbf{Top}}(\Delta^-_{top}, Y) \\
&\cong \mathtt{lim}_{\Delta^- \to X} \mathtt{sing}(Y)(-) \\
&\cong \mathtt{lim}_{\Delta^- \to X} \mathtt{Hom}_{\mathbf{sSet}}(\Delta^-, \mathtt{sing}(Y)) \\
&\cong \mathtt{Hom}_{\mathbf{sSet}}(\mathtt{colim}_{\Delta^- \to X} \Delta^-, \mathtt{sing}(Y)) \\
&\cong \mathtt{Hom}_{\mathbf{sSet}}(X, \mathtt{sing}(Y))
\end{aligned}$$

□

An n-**simplex** in a simplicial set X is a morphism $\Delta^n \to X$. By Yoneda, there is a bijection between $X[n]$ and the set of n-simplices of X. This bijection maps $\sigma : \Delta^n \to X$ to the point $\sigma_*(\mathtt{id}_{[n]})$.

Explicitly, we can write $|X|$ as a the space obtained from the disjoint union of topological simplices, an n-dimensional one for every n-simplex of X, by identifying simplices according to the morphisms in Δ (face and degeneracy relations),

$$|X| \cong \coprod_{n \in \mathbb{N}} X[n] \times \Delta_{top}[n] / \sim \ . \tag{17}$$

Here $(x, s) \in X[n] \times \Delta^n_{top}$ is identified with $(y, t) \in X[m] \times \Delta^m_{top}$, if there exists $f : [n] \to [m]$ such that $X(f)(y) = x$ and $\Delta_{top}(f)(s) = t$.

A simplex $x \in X[n]$ is called **degenerate** if there exist $i \in \{0, \ldots, n\}$ such that $x = \sigma_i(y)$ for some $y \in X[n-1]$. Let $X^{nd}[n] \subseteq X[n]$ be the set of non-degenerate simplices.

Problem 7.18. *Show that*

$$|X| \cong \coprod_{n \in \mathbb{N}} X[n]^{nd} \times \Delta^n_{top} / \sim$$

where $(x, s) \in X^{nd}[n] \times \Delta^n_{top}$ is identified with $(y, s) \in X^{nd}[n+1] \times \Delta^m_{top}$, if there exists $i \in \{0, \ldots, n+1\}$ such that $X(\partial_i)(y) = x$ and $\Delta_{top}(\partial_i)(s) = t$.

Lemma 7.19. *$|X|$ is a CW-complex.*

Proof. We use the homeomorphism of pairs

$$(D^n, S^{n-1}) \cong (\Delta^n_{top}, \partial\Delta^n_{top}) \ .$$

We consider the filtration

$$\emptyset = |X|_{-1} \subseteq |X|_0 \subseteq |X_1| \subseteq \cdots \subseteq |X| \ ,$$

where

$$|X|_k \cong \coprod_{n\in\{0,\ldots,k\}} X[n]^{nd} \times \Delta^n_{top}/\sim \ .$$

The relations are the same as in Problem 7.17. Then by construction $X \cong \mathtt{colim}_n |X|_n$. We now observe that

$$\begin{tikzcd} X^{nd}[n+1] \times \partial\Delta^{n+1}_{top} \arrow[r, "\kappa"] \arrow[d] & {|X|_n} \arrow[d] \\ X^{nd}[n+1] \times \Delta^{n+1}_{top} \arrow[r] & {|X|_{n+1}} \end{tikzcd}$$

is a push-out diagram. The attaching map κ sends a point $(x, \partial_i(t)) \in X^{nd}[n+1] \times \partial\Delta^{n+1}_{top}$ with $t \in \Delta^{n-1}_{top}$ to the point $(X(\partial_i)(x), t)$. One must check that this well-defined since the same point may have a different presentations of this form. $\square$

Example 7.20. We have $|\Delta^n| \cong \Delta^n_{top}$. This is clear from the definition, but can also be seen from the formula.

Example 7.21. We define $\partial\Delta^n \subset \Delta^n$ to be the simplicial set such that $\partial\Delta^n[m]$ is the set set of all m-simplices which are in the image of some boundary map. Then one can check that

$$|\partial\Delta^n| \cong S^{n-1} \ , \quad |\Delta^n/\partial\Delta^n| \cong S^n \ .$$

Example 7.22. For a small category $\mathbf{C}$ the space $|\mathtt{N}(\mathbf{C})|$ is called the classifying space of $\mathbf{C}$. It is known that every homotopy equivalence class of CW-complexes can be represented in the form $|\mathtt{N}(\mathbf{C})|$ for a suitable category.

Example 7.23. The unit of the adjunction $(|-|, \mathtt{sing})$ provides an natural map

$$|\mathtt{sing}(X)| \to X \ .$$

It sends the equivalence class of $(x,t) \in X[n] \times \Delta^n_{top}$ to $\Delta_{top}(x)(t)$, where we interpret x as a map of simplicial sets $\Delta^n \to X$. It is known that if X is a CW-complex, then this map is a homotopy equivalence. It always induces an isomorphism in homotopy groups.

Example 7.24. If G is a group, then $BG := |\mathbf{B}G|$ is called the **classifying space** of G. Since $|-|$ is a left-adjoint it commutes with quotients. Since $\mathbf{B}G \cong \mathbf{E}G/G$ we conclude that $BG \cong EG/G$, where $EG := |\mathbf{E}G|$. One can show that EG is contractible, that $EG \to BG$ is a locally trivial bundle, and that consequently $BG \simeq K(G,1)$. We will actually verify this in the more general context where G is a topological group.

7.3 Simplicial spaces and classifying spaces of topological groups

If G is a topological group, then the simplicial sets $\mathbf{E}G$ and $\mathbf{B}G$ defined after forgetting the topology on G are not the appropriate objects. The categories $\mathbb{E}G$ and $\mathbb{B}G$ inherit topological structures from G. In order to capture these structures we will consider category objects in **Top**, i.e. categories internal to **Top** which we call shortly topological categories. A topological category $\mathbf{C}$ has in particular topological spaces of objects $\mathtt{Ob}(\mathbf{C})$ and morphisms $\mathtt{Mor}(\mathbf{C})$. These spaces are connected by maps $s,t : \mathtt{Mor}(\mathbf{C}) \to \mathtt{Ob}(\mathbf{C})$ which determine source and target of a morphism, and a map $\mathtt{Ob}(\mathbf{C}) \to \mathtt{Mor}(\mathbf{C})$ realizing the identities.

Example 7.25. A category is a topological category with discrete spaces of objects and morphisms. A topologically enriched category is a topological category with a discrete space of objects.

Example 7.26. The category of vector subspaces of $\mathbb{R}^n$ is a topological category. Its space of objects is the disjoint union of Grassmannians

$$B := \bigsqcup_{i=0}^{n} \mathrm{Gr}(i, \mathbb{R}^n) \ .$$

Let $\xi_i \to \mathrm{Gr}(i, \mathbb{R}^n)$ be the tautological bundle. Then we define the bundle $\xi := \bigsqcup_{i=0}^{n} \xi_i \to B$. Then the space of morphisms in this topological category is given by the total space of the vector bundle

$$\mathtt{Hom}(\mathtt{pr}_1^*\xi, \mathtt{pr}_2^*\xi) \to B \times B \ .$$

Source and target are the projections to the first and second factors.

Example 7.27. The category of locally compact topological spaces $\mathbf{Top}_{lc}$ is enriched over **Top** and therefore a topological category.

Example 7.28. For a topological group G we can consider $\mathbb{B}G$ as a topological category. Its space of objects is has one point, and its space of morphisms is the group G. The composition is induced by the group structure of G and the identity section is the inclusion of the identity element.

The category $\mathbb{E}G$ is the topological category with space of objects G and space of morphisms $G \times G$. The source and target maps are given by $s : (g, h) \mapsto g$ and $t : (g, h) \mapsto gh$. The composition sends the pair of composeable morphisms $((g, h), (gh, k)$ to (g, ghk). The identity section is the diagonal $G \to G \times G$.

If $\mathbf{C}$ is a topological category, then $\mathbf{Fun}([n], \mathbf{C})$ is again a topological category.

Problem 7.29. *Show that for a topological category* $\mathbf{D}$ *whose spaces of morphisms and objects are locally compact Hausdorff spaces the category* $\mathbf{Fun}(\mathbf{D}, \mathbf{C})$ *is naturally a topological category.*

For a topological category we can refine the simplicial set

$$\mathtt{N}(\mathbf{C}) = \mathtt{Ob}\mathbf{Fun}([n], \mathbf{C})$$

to a topological space. Indeed, we can write the space of n-simplices as an iterated fibre-product

$$\mathtt{N}(\mathbf{C})[n] = \underbrace{\mathtt{Mor}(\mathbf{C}) \times_{s,\mathtt{Ob}(\mathbf{C}),t} \cdots \times_{s,\mathtt{Ob}(\mathbf{C}),t} \mathtt{Mor}(\mathbf{C})}_{n+1} .$$

Therefore the nerve of a topological category $\mathbf{C}$ is a simplicial space, i.e. $\mathtt{N}(\mathbf{C}) \in \mathbf{sTop}$.

Example 7.30. For a topological group G we get the simplicial spaces $\mathbf{B}G := \mathtt{N}(\mathbb{B}G)$ and $\mathbf{E}G := \mathtt{N}(\mathbb{E}G)$.

Example 7.31. Note that Δ^n_{top} is locally compact and Hausdorff. For a topological space X we can thus refine the simplicial set $\mathtt{sing}(X)$ to a simplicial space $\mathtt{sing}^{top}(X) \in \mathbf{sTop}$ by setting

$$\mathtt{sing}^{top}(X)[n] = \mathtt{Map}(\Delta^n_{top}, X) .$$

Lemma 7.32. *We have an adjunction*

$$| - | : \mathbf{sTop} \leftrightarrows \mathbf{Top} : \mathtt{sing}^{top} .$$

Proof. This is formal as in Lemma 7.16. □

For the following Lemma we work in the category of compactly generated weak Hausdorff spaces $\mathbf{Top}_k$. The advantage of this category is that it is cartesian closed. So the functor of taking the cartesian product with a space X is the left-adjoint in the adjunction

$$- \times X : \mathbf{Top}_k \leftrightarrows \mathbf{Top}_k : \mathtt{Map}(X, -) .$$

Being a left-adjoint the functor $-\times X$ commutes with colimits. In general, if we take the cartesian product in **Top** of two objects of $\mathbf{Top}_k$, the the result will not be compactly generated. The cartesian product in $\mathbf{Top}_k$ is obtained from the product in **Top** by modifying the topology.

The geometric realization in this context is still defined by an adjunction

$$|-|:\mathbf{sTop}_k \leftrightarrows \mathbf{Top}_k:\mathtt{sing}^{top}$$

and explicitly given by the formula (17).

Lemma 7.33. *The geometric realization $|-|:\mathbf{sTop}_k \to \mathbf{Top}_k$ preserves finite products.*

Proof. Let X, Y be simplicial spaces. By functoriality the projections

$$X \leftarrow X\times Y \to Y$$

induce maps

$$|X| \leftarrow |X\times Y| \to |Y|$$

and therefore a map

$$\zeta:|X\times Y| \to |X|\times|Y| \ .$$

We show that ζ is a homeomorphism. To this end we construct an inverse ξ. For a simplicial space X we set

$$\overline{X} := \coprod_{n\in\mathbb{N}} X[n]\times\Delta^n_{top} \ .$$

We let $\overline{X}_n$ be the component with index n. Then we have a quotient map

$$\overline{X}\to|X| \ .$$

We first define a map

$$\bar{\xi}:\overline{X}\times\overline{Y}\to|X\times Y|$$

and then observe that it factorizes over the quotient $|X|\times|Y|$.

Let

$$((x,t),(y,s))\in\overline{X}_n\times\overline{Y}_m \ .$$

Here $x\in X[n]$, $y\in Y[m]$ and $t\in\Delta^n_{top}$, $s\in\Delta^m_{top}$. We define numbers in $[0,1]$ by

$$u_k:=\sum_{i=0}^k t_i \ , \quad k=0,\dots,n-1$$

and

$$v_l:=\sum_{j=0}^l s_j \ , \quad l=0,\dots,m-1 \ .$$

Then we let

$$0 \le w_0 \le \cdots \le w_{n+m-1} \le 1$$

be obtained by ordering the numbers u_k and v_l. In addition we set $w_{-1} := 0$ and $w_{n+m} := 1$. We now define

$$r_i := w_i - w_{i-1} \ , \quad i = 0, \dots, n+m \ .$$

We consider r as a point in Δ_{top}^{n+m}.

Let

$$i_1 < \cdots < i_n \ , \quad j_1 < \cdots < j_m$$

be the disjoint sequences such that

$$w_{i_r} \in \{u_k\} \ , \forall r \ , \quad w_{j_s} \in \{v_l\} \ , \forall s \ .$$

Then

$$t = \Delta_{top}(\sigma_{j_1}) \dots \Delta_{top}(\sigma_{j_m}) r \ , \quad s = \Delta_{top}(\sigma_{i_1}) \dots \Delta_{top}(\sigma_{i_n}) r \ ,$$

Then we define

$$\bar{\xi}((x,t),(y,s)) := [(X(\sigma_{j_m}) \dots X(\sigma_{j_1})(x), Y(\sigma_{i_n}) \dots Y(\sigma_{i_1})(y)), w] \in |X \times Y| \ .$$

This map is continuous and preserves the relations for the quotient

$$\overline{X} \times \overline{Y} \to |X| \times |Y| \ .$$

Problem 7.34. *Check this assertion.*

We thus get a diagram

$$\begin{tikzcd}
\overline{X} \times \overline{Y} \arrow[r] \arrow[d] \arrow[rr, bend left, "\bar{\xi}"] & (\overline{X} \times \overline{Y})/\sim \arrow[r] \arrow[dl, dashed] & {|X \times Y|} \ . \\
{|X| \times |Y|} \arrow[urr, dotted, "\xi"] & &
\end{tikzcd}$$

In order to obtain the dotted arrow we must invert the dashed one. In general, this is not a homeomorphism since finite products do not commute with quotients. But this is true if we work in a convenient category of compactly generated weak Hausdorff spaces $\mathbf{Top}_k$.

$\square$

From now we work in the category $\mathbf{Top}_k$ without further notice. A functor between topological categories is a functor between the underlying categories with the additional property that the maps induced by the functor between the spaces of objects and morphisms are continuous. Similarly, a natural transformation between functors $\mathbf{C} \to \mathbf{D}$ between topological categories is a natural transformation between the functors on the underlying categories with the additional property that the map $\mathtt{Ob}(\mathbf{C}) \to \mathtt{Mor}(\mathbf{D})$ is continuous.

For a topological category $\mathbf{C}$ we use the abbreviation

$$|\mathbf{C}| := |\mathtt{N}(\mathbf{C})|$$

for the classifying space.

Lemma 7.35. *The following data are equivalent:*

1. *a natural transformation $h : F_0 \to F_1$ between functors $F_i : \mathbf{C} \to \mathbf{D}$ between topological categories*
2. *a functor*
$$H : [1] \times \mathbf{C} \to \mathbf{D}$$
between topological categories such that $H_{|\{i\}\times\mathbf{C}} = F_i$.

Proof. We describe the functor H determined by h. On objects we define

$$H((0, C)) := F_0(C) \ , \quad H((1, C)) := F_1(C) \ .$$

On morphisms we set

$$H(\mathtt{id}_i, f) := F_i(f) \ , \quad i = 0, 1$$

and

$$H(0 \to 1, \mathtt{id}_C) := h_C \ .$$

One checks that this prescription determines H uniquely. Vice versa, from these formulas we see how to read off h from H. □

There is a natural definition of the notion of an equivalence between topological categories.

Lemma 7.36. *If $F : \mathbf{C} \to \mathbf{D}$ be an equivalence between topological categories, then $|F| : |\mathbf{C}| \to |\mathbf{D}|$ is a homotopy equivalence.*

Proof. Let $G : \mathbf{D} \to \mathbf{C}$ be a functor and $\phi : \mathtt{id}_{\mathbf{C}} \to G \circ F$ and $\mathtt{id}_{\mathbf{D}} \to F \circ G$ be natural isomorphisms. Using Lemma 7.34 from ϕ we get a functor $H : [1] \times \mathbf{C} \to \mathbf{C}$ which restrict to $\mathtt{id}_{\mathbf{C}}$ and $G \circ F$ at 0 and 1. Note that $|[1]| \cong [0, 1]$. Using Lemma 7.32 and the fact that $\mathtt{N}$ preserves products we get a homotopy

$$|H| : |[0, 1]| \times |\mathbf{C}| \to \mathbf{C}|$$

from $\mathtt{id}_{|\mathbf{C}|}$ to $|G| \circ |F|$. We argue similarly for $|F| \circ |G|$. □

For a topological group we now consider the spaces $EG := |\mathbf{B}G|$ and $BG := |\mathbf{B}G|$.

We call a space X **locally contractible** if every point $x \in X$ admits a basis of neighbourhoods which are contractible to the point x. For example, a manifold is locally contractible. If G is a group, then it suffices to show that the identity element admits a basis of neighbourhoods which can be contracted to this element.

Lemma 7.37. *1. EG is a contractible space.*

2. G acts freely and properly on EG with quotient BG

3. If G is locally contractible, then $EG \to BG$ is a locally trivial bundle with fibre G.

Proof. The space EG is the classifying space of the topological category described in Example 7.27. Let $*$ be the terminal category. Let $A : * \to \mathbb{E}G$ be the functor which sends the unique object of $*$ to $1 \in G$. We define $B : \mathbb{E}G \to *$ in the canonical way. Then $B \circ A = \mathtt{id}_*$. We construct a natural transformation

$$h : A \circ B \to \mathtt{id}_{\mathbb{E}G} \ , \quad h : G \to G \times G \ , \quad h(g) := (e, g) \ .$$

Problem 7.38. *Check the details.*

We see that $|A| : EG \to *$ and $|B| : * \to EG$ are inverse to each other homotopy equivalences. This implies that EG is contractible.

We consider the group G as a group $\mathbf{G}$ in **sTop** by setting $\mathbf{G}[n] := G$ for all $n \in \mathbb{N}$ and defining all structure maps to be the identity. Thus $\mathbf{G}$ is the **constant simplicial space** on G. The group $\mathbf{G}$ acts on $\mathbf{E}G$ as follows:

$$\mathbf{E}G[n] \times \mathbf{G}[n] \mapsto \mathbf{E}G[n] \ , \quad ((g_0, \ldots, g_n), h) \mapsto (g_0 h, \ldots, g_n h) \ .$$

We define a map $\mathbf{E}G \to \mathbf{BG}$ by

$$(g_0, \ldots, g_n) \mapsto (g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{n-1} g_n^{-1}) \ .$$

Problem 7.39. *Check that this map is well-defined.*

We argue that this map presents $\mathbf{BG}$ as the quotient $\mathbf{E}G/\mathbf{G}$. For every $n \in \mathbb{N}$ we have an isomorphism of G-spaces $\mathbf{BG}[n] \times G \to \mathbf{E}G[n]$ over $\mathbf{BG}[n]$ which sends

$$((g_1, \ldots, g_n), h) \to (h, g_1^{-1} h, g_2^{-1} g_1^{-1} h, \ldots, g_n^{-1} \ldots g_1^{-1} h) \ . \tag{18}$$

Note that this morphism is not compactible with the simplicial structures.

Let X be a simplicial space and $\mathbf{E}G \to X$ a $\mathbf{G}$-invariant morphism. Then for every $n \in \mathbb{N}$ we have the G-invariant continuous map $\mathbf{E}G[n] \to X[n]$. By the discussion above it determines a unique map $\mathbf{BG}[n] \to X$. It follows from the uniqueness assertion that the collection of these factorizations for all $n \in \mathbb{N}$ together give a morphism of simplicial spaces $\mathbf{BG} \to X$ which is unique.

We now observe that $G \cong |\mathbf{G}|$ since in the constant simplicial group $\mathbf{G}$ all higher simplices are degenerated. Since $|-|$ preserves products it sends the action of $\mathbf{G}$ on $\mathbf{E}G$ to the action of G on EG. Since the action of G on $\mathbf{B}G[n]$ is proper and free one checks that the action of G on EG is proper and free, too.

As a left-adjoint the geometric realization functor $|-|$ preserves colimits, so in particular coequalizers. We conclude that $BG \cong EG/G$.

It remains to construct local sections. We use the filtration

$$\emptyset = BG_{-1} \subseteq BG_0 \subseteq BG_1 \subseteq \cdots \subseteq BG \ ,$$

where BG_n is the image of

$$\bigsqcup_{k=0}^{n} \mathbf{B}G[k] \times \Delta^k_{top} \ .$$

Let $x \in BG$. Then there exists a smalles k_0 such that $x \in BG_{k_0}$. Then $x = [a, t]$, where $a \in \mathbf{BG}[k_0]$ and $t \in \mathrm{Int}(\Delta^{k_0}_{top})$. For every $k \in \mathbb{N}$ the pull-back

$$(\mathbf{BG}[k] \times \Delta^k_{top}) \times_{BG} EG \to \mathbf{BG}[k] \times \Delta^k_{top}$$

is trivial (use 18). Hence we can find a section over the open subset

$$\mathbf{BG}[k_0] \times \mathrm{Int}(\Delta^{k_0}_{top}) \subset BG_{k_0} \ .$$

We now extend this section inductively. We decompose the space of k-simplices into the degenerated and non-degeneted part:

$$\mathbf{BG}[k] := \mathbf{BG}[k]^{nd} \sqcup \mathbf{BG}[k]^d \ .$$

Here $\mathbf{BG}[k]^d$ is the subspace of all tuples $(g_1, \ldots, g_k) \in \mathbf{BG}[k]$ where $g_i = 1$ for at least one index $i \in \{1, \ldots, k\}$. Then we have

$$BG_k = BG_{k-1} \cup_{\mathbf{BG}[k]^d \times \Delta^k_{top} \cup \mathbf{BG}[k]^{nd} \times \partial\Delta^k_{top}} \mathbf{BG}[k] \times \Delta^k_{top} \ .$$

We now assume that we have a section defined on an open neighbourhood $V \subseteq BG_{k-1}$ of a point $x \in BG_{k-1}$. Then we must extend it to an open neighbourhood of x considered as a point in BG_k. Using the trivialization of the pull-back of the bundle to $\mathbf{BG}[k] \times \Delta^k_{top}$ we can identify the data of the section with a map

$$\mathbf{BG}[k]^d \times \Delta^k_{top} \cup \mathbf{BG}[k]^{nd} \times \partial\Delta^k_{top} \supseteq V \to G \ .$$

We now use that

$$\mathbf{BG}[k]^d \times \Delta^k_{top} \cup \mathbf{BG}[k]^{nd} \times \partial\Delta^k_{top} \hookrightarrow \mathbf{BG}[k] \times \Delta^k_{top}$$

is a neighbourhood deformation retract.

Problem 7.40. *Deduce this fact from the property that G is locally contractible.*

Hence we can extend the section as required. $\square$

7.4 Classification of principal bundles

Let G be a topological group.

Definition 7.41. *A G-**principal bundle** is a locally trivial bundle $E \to B$ where E admits a fibrewise right action such that the natural map*

$$E \times G \to E \times_B E \ , (e, g) \mapsto (e, eg)$$

is a homeomorphism.

Problem 7.42. *Let $E \to B$ be a map between topological spaces admitting local sections and assume that G acts fibrewise on E such that $E \times G \to E \times_B E$ is a homeomorphism. Then $E \to B$ is a G-principal bundle.*

The problem thus consists in translating the condition of being local trivial in the condition of having local sections.

Assume that $E \to B$ is trivial. Then choosing a section s we get a G-equivariant homeomorphism

$$B \times G \stackrel{\simeq}{\to} E \ , \quad (b, g) \mapsto s(b)g$$

over B. So we see that the fibre of a G-principal bundle is isomorphic to G as a right G-space.

Example 7.43. For every spaces B and group G we have the trivial G-principal bundle $B \times G \to B$.

Example 7.44. The map $EG \to BG$ is a G-principal bundle. It is often called the **universal G-principal bundle**. We shall see below, why.

Example 7.45. If $V \to B$ is an n-dimensional vector bundle, then we can form the space $\mathtt{Fr}(V)$ of pairs (b, ϕ), where $b \in B$ and $\phi : \mathbb{R}^n \to V_b$ is an isomorphism. The group $GL(n, \mathbb{R})$ acts on $\mathtt{Fr}(V)$ from the right by $(b, \phi)g := (b, \phi \circ g)$. We have a projection $\mathtt{Fr}(V) \to B$ which sends (b, ϕ) to b. This describes a $GL(n, \mathbb{R})$-principal bundle called the **frame bundle** of $V \to B$.

Example 7.46. Let F be a locally compact Hausdorff space. Then we have a topological group $\mathtt{Aut}(F) \subset \mathtt{Map}(F, F)$. If $E \to B$ is a locally trivial fibre bundle with fibre F, then we can generalize the construction of the frame bundle. We let $\mathtt{Fr}(E)$ be the space of pairs (b, ϕ), where $b \in B$ and $\phi : F \to E_b$ is a homeomorphism. The action of $\mathtt{Aut}(F)$ on E is given by $(b, \phi)g := (b, \phi \circ g)$. Then $\mathtt{Fr}(E) \to B$ is a $\mathtt{Aut}(F)$-principal bundle.

Example 7.47. Let $\tilde{X} \to X$ be a Galois covering with group Γ, see Subsection 1.7. This is a Γ-principal bundle.

Let $E \to B$ be a G-principal bundle and $f : A \to B$ be a continuous map. Then

$$\mathtt{pr}_A : A \times_B E \to A$$

is again a G-principal bundle. It is called the pull-back and is often denoted by $f^*E \to A$.

An isomorphism between G-principal bundles $E \to B$ and $F \to B$ over the same space B is a G-invariant map over B.

Problem 7.48. *Show that such a map is automatically invertible.*

So we do not have to require invertibility explicitly.

Let $E \to B$ and $E' \to B'$ be G-principal bundles.

Lemma 7.49. *A square*

$$\begin{tikzcd} E' \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

where F is G-equivariant is a pull-back square.

Proof. We have a canonical map $E' \to B' \times_B E$ which is G-equivariant and over B' and hence a homeomorphism. □

Let F be a right G-space and $E \to G$ be a G-principal bundle. Then we form the map

$$\pi : (E \times F)/G \to B \ .$$

The map π is the projection of a locally trivial fibre bundle with fibre F. It is called the **associated fibre bundle** to the G-principal bundle $E \to B$ and the G-space F.

Lemma 7.50. *There is a bijection between sections of the associated fibre bundle π and G-equivariant maps $E \to F$.*

Proof. Let s be a section of π. Then we define the G-equivariant map $\sigma : E \to F$ by $\sigma(e) := f$, where $f \in F$ is uniquely determined by the condition that $s(b) = [e, f]$. It is obvious from this definition that $\sigma(eg) = fg$.

If $\sigma : E \to F$ is a G-equivariant map, then we define the section s by $s(b) := [e, \sigma(e)]$ for any choice of $e \in E_b$. The value $s(b)$ is well-defined. Indeed, if we would chose eg instead of e, then $[eg, \sigma(eg)] = [eg, \sigma(e)g] = [e, \sigma(e)]$. $\square$

Let $E \to B$ be a G-principal bundle.

Lemma 7.51. *There is a bijection between the sets of trivializations of the bundle and of sections.*

Proof. Given a trivialization $\phi : E \xrightarrow{\cong} B \times G$ we have the section $b \mapsto \phi^{-1}((b, 1))$. Given a section s we define a trivialization by $\phi(e) := (\pi(b), e^{-1}(\phi)s(\pi(e)))$. Here for points $e, f \in E_b$ we let $e^{-1}f \in G$ be defined such that $e(e^{-1}f) = f$. $\square$

Example 7.52. Let $\pi : E \to B$ be a G-principal bundle. Then $E \times_B E \to E$ has a canonical trivialization. It corresponds to the section $\mathtt{diag} : E \to E \times_B E$.

We now fix a topological group. We consider the functor

$$P_G : \mathbf{Top}^{op} \to \mathbf{Set}$$

which sends the space $B \in \mathbf{Top}$ to the set $P_G(B)$ of isomorphism classes of G-principal bundles $[E \to B]$ on B and a continuous map $f : A \to B$ to the map

$$f^* : P_G(B) \to P_G(A) \ , \quad [E \to B] \mapsto [A \times_B E \to A] \ .$$

Remark 7.53. If $E \to B$ is a G-principal bundle, then $B \cong E/G$.

In the following we repeatedly assume that base spaces are paracompact Hausdorff. The reason is that this assumption appears in Theorem 5.22. We need the fact, that a locally trivial bundle over a paracompact Hausdorff space is a fibration. In particular we will use the homotopy extension property for sections.

Alternatively we can assume that the base spaces are CW-complexes. Then the proof of the homotopy extension property for sections is easier.

Let $\mathbf{Top}_{ph} \subset \mathbf{Top}$ denote the subcategory of paracompact Hausdorff spaces.

Lemma 7.54. *Assume that $G \in \mathbf{Top}_{ph}$. Then the restriction of the functor P_G to $\mathbf{Top}_{ph}$ is homotopy invariant.*

Proof. Let $E \to I \times B$ be a G-principal bundle over $B \in \mathbf{Top}_{ph}$. We must show that there is an isomorphism $E_0 \cong E_1$ of G-principal bundles, where $E_i \to B$ denote the restrictions to the endpoints of the interval for $i = 0, 1$.

We consider the bundle
$$\pi : (E \times E_0)/G \to I \times B \ .$$
The identity of E_0 is a section of π over $\{0\} \times B$. Since π is a fibration this section extends to a section s defined on all of $I \times B$. It corresponds to a G-equivariant map $E \to E_0$ over an induced map $H : I \times B \to B$ obtained by passing to quotients. Let $s_1 := s_{\{1\} \times B}$. We now consider the lifting problem

$$\begin{tikzcd} \{1\} \times B \arrow[r, "s_1"] \arrow[d] & (E_1 \times E_0)/G \arrow[d, "p"] \\ I \times B \arrow[r, "H"] \arrow[ur, dotted] & B \end{tikzcd} \ .$$

The lift exists again since p is a fibration. Note that $H_{|\{0\} \times B} = \mathtt{id}_B$. The evaluation of the lift at $\{0\} \times B$ is a section which corresponds to a G-equivariant map $E_1 \to E_0$ over B, i.e an isomorphism of G-principal bundles $E_1 \cong E_0$. □

□

We have a natural transformation of functors $\mathbf{Top} \to \mathbf{Set}$
$$\Phi : [-, BG] \to P_G(-) \ , \quad \Phi_B : [f : B \to BG] \mapsto [f^*EG \to B] \ .$$

Proposition 7.55. *If $B \in \mathbf{Top}_{ph}$, then the map Φ_B is a bijection.*

Proof. We first show that Φ is surjective. Let $E \to B$ be a G-principal bundle. We must find a map $f : B \to BG$ such that $\Phi_B([f]) = [E \to B]$. We consider the associated fibre bundle $\pi : (E \times EG)/G \to B$ for the G-space EG. This is a locally trivial fibre bundle with fibre EG. Since we assume that $B \in \mathbf{Top}_{ph}$ the map π is a fibration. A fibration with a contractible fibre is a homotopy equivalence. Let $s_0 : B \to (E \times EG)/G$ be an inverse up to homotopy. Again using the fibration property we can modify this map by a homotopy such that it becomes a section s. It may be obtained as the evaluation at $\{1\} \times B$ of a lifting in
$$\begin{tikzcd} \{0\} \times B \arrow[r, "s_0"] \arrow[d] & E \arrow[d] \\ I \times B \arrow[r] \arrow[ur, dotted] & B \end{tikzcd} \ ,$$
where the lower horizontal map is the homotopy from $\pi \circ s_0$ to $\mathtt{id}_B$. The section s is by Lemma 7.49 equivalent to a G-equivariant map $E \to E \times EG$. Its second component is

a G-equivariant map $E \to EG$. By passing to quotients we get a map $f : B \to BG$. We now apply Lemma 7.48 to deduce that the resulting square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r] & BG \end{tikzcd}$$

is a pull-back square. Therefore $[E \to B] = \Phi_B([f])$. This shows the surjectivity of Φ.

We now show injectivity. We consider two maps $f, g : B \to BG$ and assume that $\Phi_B([f]) = \Phi_B([g])$. We must show that then f and g are homotopic. By assumption we have an isomorphism of G-principal bundles $\psi : f^*EG \to g^*EG$. We define the total space of a G-principal bundle $E \to I \times B$ as the push-out

$$\begin{tikzcd} (-1/2,1/2) \times f^*EG \arrow[r, "incl\times\psi"] \arrow[d, "incl\times\mathtt{id}"] & (-1/2,1] \times g^*EG \arrow[d] \\ {[-1,1/2) \times f^*EG} \arrow[r] & E \end{tikzcd} .$$

We now consider the associated bundle $(E \times EG)/G \to I \times B$. The canonical map $f^*EG \to EG$ induces a section of this associated bundle restricted to $\{-1\} \times B$. Similarly we get a section over $\{1\} \times B$ from $g^*E \to EG$. Since the fibre of the bundle is contractible, we can solve the lifting problem in

$$\begin{tikzcd} \{0,1\} \times B \arrow[r] \arrow[d] & (E \times EG)/G \arrow[d] \\ I \times B \arrow[r, equal] \arrow[ur, dotted] & I \times B \end{tikzcd} .$$

Here we proceed in two steps. In the first

The lift can be interpreted as a G-equivariant map $E \to EG$.

By passing to quotients we get the homotopy $H : I \times B \to BG$ from f to g. $\square$

Remark 7.56. Observe that in this argument we have only used that $EG \to BG$ is a G-principal bundle such that EG is contractible. Let $E \to B$ a G-principal bundle over a space $B \in \mathbf{Top}_{ph}$ such that E is contractible. Then we get an isomorphism of functors $[-, B] \cong [-, BG]$. From this we conclude that B and BG are homotopy equivalent.

A G-principal bundle $E \to B$ over a base space $B \in \mathbf{Top}_{ph}$ with contractible total space E is called **a universal G-bundle**. The base space of a universal G-bundle is homotopy equivalent to BG.

From the long exact sequence in homotopy we get isomorphisms

$$\pi_{n+1}(BG) \cong \pi_n(G) \tag{19}$$

for all $n \geq 0$.

Example 7.57. We consider the $U(1)$-principal bundles $S^{2n-1} \to \mathbb{CP}^n$. There are inclusions

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & S^{2(n+1)-1} \arrow[d] \\ \mathbb{CP}^n \arrow[r] & \mathbb{CP}^{n+1} \end{tikzcd} .$$

If we form the colimit, then we get a $U(1)$-bundle

$$S^\infty \to \mathbb{CP}^\infty \ .$$

Problem 7.58. *Verify that this bundle has local sections.*

Since S^∞ is contractible, this is the universal $U(1)$-bundle.

Problem 7.59. *Show that* $S^\infty := \mathtt{colim}_{n\in\mathbb{N}} S^n$ *is contractible.*

Example 7.60. We consider the group $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ with quotient T^n. Then $\mathbb{R}^n \to T^n$ is a $\mathbb{Z}^n$-principal bundle. Since $\mathbb{R}^n$ we see that $T^n \simeq B\mathbb{Z}^n$.

Example 7.61. We have a bijection between isomorphism classes of n-dimensional real vector bundles on a space B and $GL(n,\mathbb{R})$-principal bundles. This bijection is given as follows:

1. If $V \to B$ is an n-dimensional real vector bundle, then the corresponding $GL(n,\mathbb{R})$-principal bundle is the frame bundle $\mathtt{Fr}(V) \to B$, see Example 7.44.
2. If $E \to B$ is a G-principal bundle, then we let $V := E \times_{GL(n,\mathbb{R})} \mathbb{R}^n \to B$ be the associated vector bundle.

We have the isomorphisms

$$\mathtt{Fr}(V) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \to V \ , \quad ((b,\phi),x) \mapsto \phi(x)$$

and

$$E \to \mathtt{Fr}(E \times_{GL(n,\mathbb{R})} \mathbb{R}^n) \ , \quad e \mapsto (\pi(e), (x \mapsto [e,x])) \ .$$

Hence, for $B \in \mathbf{Top}_{ph}$ we can identify the set $[B, BGL(n,\mathbb{R})]$ also with the set of isomorphism classes of n-dimensional real vector bundles over B.

7.5 Topological abelian groups

For a based space B we consider the path fibration defined by the pull-back

$$\begin{tikzcd} PB \arrow[r] \arrow[d] & B^I \arrow[d, "(\mathtt{ev}_0,\mathtt{ev}_1)"] \\ B \arrow[r, "*_B\times\mathtt{id}"] & BG\times BG \end{tikzcd} .$$

Lemma 7.62. *The space PB is contractible.*

Proof. A point in PB is a path $(t\mapsto\gamma(t))$ such that $\gamma(0)=*$. The base point $*_{PB}$ of PB is the constant path with value $*_B$. Using this notation the contraction is given by

$$H_{PB}: I\times PB\to PB\ ,(s,(t\mapsto\gamma(t)))\mapsto(t\to\gamma(st))\ .$$

□

Let now $f:E\to B$ be a fibration such that E is contractible and $F:=f^{-1}(*_B)$ be the fibre over the base point of B.

Lemma 7.63. *We have a homotopy equivalence $F\simeq\Omega B$.*

Proof. We consider a diagram

$$\begin{tikzcd} F \arrow[r, dotted, "\kappa"] \arrow[d] & \Omega B \arrow[d] \\ E \arrow[r, dotted, "k"] \arrow[d, "f"] & PB \arrow[d, "\mathtt{ev}_1"] \\ B \arrow[r, equal] & B \end{tikzcd} .$$

The choice of a contracting homotopy of $H:I\times E\to E$ from $\mathtt{id}$ to $\mathtt{const}_{*_E}$ induces the map k by

$$k(e)=(t\mapsto H(t,e))\ .$$

The map κ is obtained from k by restriction to the fibre.

We now show that κ is a homotopy equivalence. We solve the homotopy extension problem for sections

$$\begin{tikzcd} \{0\}\times PB \arrow[r, "\mathtt{const}_{*_E}"] \arrow[d, "ev_1"] & E \arrow[d, "f"] \\ I\times PB \arrow[r, "f\circ H_{PB}"] \arrow[ur, dotted] & B \end{tikzcd}$$

The restriction of the lift to $\{1\}\times PB$ yields a fibrewise map $\ell: PB\to E$. We now show that the restriction $\lambda:\Omega B\to F$ of ℓ is a homotopy inverse of κ.

We clearly have a homotopy $h : I \times E \to E$ from $\ell \circ k$ to $\mathtt{id}_E$, but which is not necessarily fibrewise. We consider a map

$$(\{0,1\} \times I \cup I \times \{1\}) \times E \to B$$

which is the composition of the projection to E with f on $\{0,1\} \times I \times E$ and equal to the loop $\gamma : (t,e) \mapsto f(h(t,e))$ on $I \times \{1\} \times E$. We can extend this map to a map $I \times I \times E \to B$ which is the composition of the projection to E with f on $I \times \{0\} \times E$. To see this let $H_E : I \times E \to E$ be a contracting homotopy of E and consider

$$S^1 \times I \times E \ni (t,s,e) \mapsto f(h(t, H_E(s,e))) \in B \ .$$

We have

$$f(h(t, H_E(1,e))) = \gamma(t,e) \ .$$

The loop $(t \to h(t, H_E(0,e)))$ is independent of e. Since E is contractible we can extend the restriction of the above map to $S^1 \times \{0\} \times E$ over $D^2 \times E$. We now identify $(S^1 \times I) \cup_{S^1 \times \{0\}} D^2 \cong D^2$ (we fill an annulus by a disc).

Using the claim we can consider the homotopy extension problem for a lift

$$\begin{tikzcd}
(\{0,1\} \times I \cup I \times \{1\}) \times E \arrow[r, "\tilde{h}"] \arrow[d] & E \arrow[d, "f"] \\
I^2 \times E \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd} \quad ,$$

where

$$\tilde{h}(0,t,e) := (\lambda \circ \kappa)(e) \ , \quad \tilde{h}(1,t,e) := e \ , \quad \tilde{h}(s,1,e) := h(s,e) \ .$$

The evaluation of the lift at $I \times \{0\}$ is the required homotopy between $\lambda \circ \kappa$ and $\mathtt{id}_F$.

The argument for $\kappa \circ \lambda$ is similar. □

Remark 7.64. This Lemma is a standard method to identify a space F with the loop space of another space B. To this end must embed into a fibration $E \to B$ with contractible total space E.

Corollary 7.65. *Assume that $E \to B$ is a universal G-principal bundle. Then we have a homotopy equivalence*

$$G \simeq \Omega G \ .$$

In particular, $G \simeq \Omega BG$.

Proof. Apply Lemma 7.62 to the fibration $EG \to BG$. □

Problem 7.66. *Study the compatibility of the group structures.*

Lemma 7.67. *If G is an abelian topological group, then $\mathbf{B}G$ is an abelian group in* **sTop** *and hence BG is an abelian topological group.*

Proof. We observe that $\mathbf{B}G$ is an abelian group in **sTop**. The operation $\mathbf{B}G\times\mathbf{B}G \to \mathbf{B}G$ is given on the level of n-simplices given by

$$(g_1,\dots,g_n)+(h_1,\dots,h_n):=(g_1+h_1,\dots,g_n+h_n)\ .$$

Problem 7.68. *Analyse where the condition that G is abelian is important.-*

Since the geometric realization preserves products $|-|$ we conclude that $BG=|\mathbf{B}G|$ is an abelian group in **Top**.

Remark 7.69. For example, the product in BG is given by

$$BG\times BG\cong|\mathbf{B}G|\times|\mathbf{B}G|\cong|\mathbf{B}G\times\mathbf{B}G|\stackrel{|+|}{\to}|\mathbf{B}G|=BG\ .$$

We interpret the unit of G as a homomorphism $1\to G$, where 1 is thre trivial group. Then we get a map $\mathbf{B}1\to\mathbf{B}G$. Using that $B1\cong|\mathbf{B}1|\cong *$ we get a map

$$*\cong B1\to BG$$

which represents the unit of BG.

Remark 7.70. More formally we can observe that the topological category $\mathbb{B}G$ is symmetric monoidal where the tensor product of morphisms is given by the sum. Thus $\mathbb{B}G$ is an abelian group object in topological catgeories and therefore $\mathbf{B}G$ an abelian group object in **sTop**. Finally we get the topological abelian group BG.

If G is a topological abelian group, then BG is again a topological abelian group. Therefore we can iterate the construction of the classifying space.

Definition 7.71. *Let G be a topological abelian group. For $n\in\mathbb{N}$ we define inductively $B^nG:=B(B^{n-1}G)$.*

Let G be a discrete group.

Lemma 7.72. *The space B^nG has the weak homotopy type of $K(G,n)$.*

Proof. Iterating (19) we get

$$\pi_i(B^nG)\cong\pi_{i-n}(G)\cong\left\{\begin{array}{cc}G & i=n\\ 0 & else\end{array}\right.$$

□

Corollary 7.64 implies inductively that

$$B^nG\to\Omega B^{n+1}G$$

is a homotopy equivalence.

Corollary 7.73. *For every topological abelian group G we get a spectrum*

$$\mathbf{E} = ((E_n)_{n\in\mathbb{Z}}, (\sigma_n)_{n\in\mathbb{Z}}) \ .$$

For $n \in \mathbb{N}$ we set

$$E_n := B^n G \ , \quad B^n G \to \Omega B^{n+1} G \ .$$

For $n \in \mathbb{Z}$ and $n < 0$ we set $E_n := \Omega^{-n} E_0$ und use the canonical structure maps.

If we apply this construction to the discrete abelian group A, then we get the **Eilenberg-MacLane spectrum H**A.

7.6 Homology

Let

$$\mathbf{E} = ((E_n)_{n\in\mathbb{Z}}, (\sigma_n)_{n\in\mathbb{Z}})$$

be a spectrum. The cohomology $H^*(X;\mathbf{E})$ of a pointed space X with coefficients in $\mathbf{E}$ has been defined in Definition 4.2. In particular the homotopy groups of the spectrum $\mathbf{E}$ are defined by

$$\pi_n^s(\mathbf{E}) := E^{-n} := H^0(S^n;\mathbf{E}) \ , \quad n \in \mathbb{Z} \tag{20}$$

In this section we consider the dual theory, the homology $H_*(X;\mathbf{E})$ of a pointed space X with coefficients in $\mathbf{E}$.

In order to define the homology as a functor we must consider a category of spectra. A morphism between spectra $f : \mathbf{E} \to \mathbf{F}$ is a family of maps $f_n : E_n \to F_n$ for all $n \in \mathbb{Z}$ which preserve the structure maps, i.e. the diagrams

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n^{\mathbf{E}}"] & \Sigma F_n \arrow[d, "\sigma_n^{\mathbf{F}}"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & F_{n+1} \end{tikzcd}$$

commute. A morphism between prespectra is defined similarly. In Definition 7.2 we have defined the spectrum $R(\mathbf{E})$ associated to a prespectrum $\mathbf{E}$. This construction provides a functor

$$R : \mathit{prespectra} \to \mathit{spectra} \ . \tag{21}$$

Given a prespectrum $\mathbf{E}$ now define a functor from pointed spaces to prespectra

$$\mathbf{Top}_* \to \mathit{prespectra} \tag{22}$$

by the following description: It sends the pointed space X to the prespectrum

$$\mathbb{N} \ni n \mapsto X \wedge E_n \ , \quad \Sigma(X \wedge E_n) \cong X \wedge \Sigma E_n \stackrel{\mathtt{id}_X \wedge \sigma_n}{\to} X \wedge E_{n+1} \ .$$

On morphisms this functor is defined in the obvious way. We now obtain a functor

$$- \wedge \mathbf{E} : \mathbf{Top}_* \to \mathit{spectra}$$

from pointed spaces to spectra by post-composing the functor (22) with the associated spectrum construction (21). Recall $\pi_k^s(-)$ from (20).

Definition 7.74. *We define the homology of X with coefficients in $\mathbf{E}$ by*

$$H_k(X;\mathbf{E}) := \pi_k^s(X \wedge \mathbf{E}) \ , \quad k \in \mathbb{Z}$$

By Lemma 7.5 we can express the homology of X with coefficients in $\mathbf{E}$ directly in terms of X and the constituents E_n of $\mathbf{E}$. If X is compact, then we have

$$H_k(X;\mathbf{E}) \cong \mathtt{colim}_n \pi_n(X \wedge E_{n-k}) \ . \tag{23}$$

In the following we state the obvious properties of homology:

1. The construction $X \mapsto X \wedge \mathbf{E}$ preserves colimits. So in particular, if (X_α) is a diagram of compact spaces, then

   $$H_k(\mathtt{colim}_\alpha X_\alpha;\mathbf{E}) \to \mathtt{colim}_\alpha H_k(X_\alpha;\mathbf{E})$$

   is an isomorphism. We see that (23) holds true for general spaces whose topology is compactly generated.

2. $X \mapsto H_k(X;\mathbf{E})$ is a homotopy invariant functor from $\mathbf{Top}_*^{op}$ to abelian groups. On compactly generated spaces this is most easily seen from (23) since the functors $X \mapsto \pi_n(X \wedge E_{n-k})$ are homotopy invariant for all $n \in \mathbb{N}$ and take values in abelian groups form $n \geq 2$.

3. Homology has a suspension isomorphism

   $$H_k(\Sigma X;\mathbf{E}) \cong H_{k-1}(X;\mathbf{E}) \ .$$

   Indeed, again using (23), we calculate

   $$\begin{aligned} H_k(\Sigma X;\mathbf{E}) &\cong \mathtt{colim}_n \pi_n(\Sigma X \wedge E_{n-k}) \\ &\cong \mathtt{colim}_n \pi_n(X \wedge E_{n-k+1}) \\ &\cong H_{k+1}(X;\mathbf{E}) \end{aligned}$$

4. The homology of spheres is given by

   $$H_k(S^n;\mathbf{E}) \cong \pi_{k+n}(\mathbf{E}) \ .$$

Next we consider the long exact sequence of a pair and its generalization. Compared with the same feature for cohomology the case of homology is more complicated. For a map $f : A \to X$ we consider the mapping cone sequence Cor. 3.25 in pointed spaces

$$A \to X \to C(f) \to \Sigma A \to \Sigma X \to \dots \ . \tag{24}$$

It immediately follows from the construction (14) of $R(-)$ that

$$\Sigma X \wedge \mathbf{E} \cong \Sigma(X \wedge \mathbf{E}) \ ,$$

see Example 4.10 for the definition of the shift of a spectrum. We get an induced sequence of maps spectra

$$A \wedge \mathbf{E} \to X \wedge \mathbf{E} \to C(f) \wedge \mathbf{E} \to \Sigma(A \wedge \mathbf{E}) \ . \tag{25}$$

Since Σ is an isomorphism this sequence can be prolonged to the left and right.

Proposition 7.75. *The sequence* (25) *induces a long exact sequence in homology*

$$\dots \to H_k(A;\mathbf{E}) \to H_k(X;\mathbf{E}) \to H_k(C(f);\mathbf{E}) \to H_{k-1}(A;\mathbf{E}) \to \dots$$

Proof. We use the suspension isomorphism for $\ell \geq 0$

$$H_k(X;\mathbf{E}) \cong H_{k+\ell}(\Sigma^\ell X;\mathbf{E}) \ .$$

The pair of pointed spaces $(\Sigma^\ell X \wedge E_{n-k-\ell}, \Sigma^\ell A \wedge E_{n-k-\ell})$ gives rise to an exact sequence

$$\pi_n(\Sigma^\ell A \wedge E_{n-k-\ell}) \to \pi_n(\Sigma^\ell X \wedge E_{n-k-\ell}) \to \pi_n(\Sigma^\ell X \wedge E_{n-k}, \Sigma^\ell A \wedge E_{n-k-\ell})$$

of homotopy groups. An ℓ-fold suspension of a space is $\ell-1$-connected by Lemma 6.25. We further use the identity $(\Sigma X) \wedge Y \cong \Sigma(X \wedge Y)$. From Theorem 6.23 we get the isomorphism

$$\pi_n(\Sigma^\ell X \wedge E_{n-k}, \Sigma^\ell A \wedge E_{n-k-\ell}) \cong \pi_n(\Sigma^\ell C(f) \wedge E_{n-k-\ell})$$

for all $n \leq 2\ell - 3$. We define $\ell(n)$ to be the smallest integer greater $\frac{n+3}{2}$. Then by (23) we have

$$H_k(X;\mathbf{E}) \cong \mathtt{colim}_n \pi_n(\Sigma^{\ell(n)} X \wedge E_{n-k-\ell(n)}) \ .$$

The exactness of

$$\pi_n(\Sigma^{\ell(n)} A \wedge E_{n-k-\ell(n)}) \to \pi_n(\Sigma^{\ell(n)} X \wedge E_{n-k-\ell(n)}) \to \pi_n(\Sigma^{\ell(n)} C(f) \wedge E_{n-k-\ell(n)})$$

implies that

$$H_k(A;\mathbf{E}) \to H_k(X;\mathbf{E}) \to H_k(C(f);\mathbf{E})$$

is exact. It extends further to a long exact sequence using Corollary 3.25. □

Remark 7.76. We consider a sequence of maps between spectra

$$\mathbf{E} \to \mathbf{F} \to \mathbf{G} \to \Sigma\mathbf{E} \ . \tag{26}$$

The choice of a homotopy of the composition $F_n \to G_n \to E_{n+1}$ to a constant map is equivalent to the data of the following diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & \Omega E_{n+1} \arrow[d] \\ F_n \arrow[r] \arrow[d] & \mathrm{Fib}(G_n \to E_{n+1}) \arrow[d] \\ G_n \arrow[r, equal] \arrow[d] & G_n \arrow[d] \\ E_{n+1} \arrow[r, equal] & E_{n+1} \end{tikzcd} \tag{27}$$

(see 5.3 for a Definition of the homotopy fibre).

Definition 7.77. *We say that the sequence of maps between spectra* (26) *is a* ***fibre sequence***, *if the sequences of spaces*

$$E_n \to F_n \to G_n \to E_{n+1}$$

for all $n \in \mathbb{N}$ fit into diagrams (26) *in which the horizontal maps are homotopy equivalences and the upper horizontal map $E_n \to \Omega E_{n+1}$ is homotopic to the structure map of* $\mathbf{E}$.

Lemma 7.78. *If* (26) *is a fibre sequence of spectra, then we have a long exact sequence*

$$\cdots \to \pi^s_n(\mathbf{E}) \to \pi^s_n(\mathbf{F}) \to \pi^s_n(\mathbf{G}) \to \pi^s_{n-1}(\mathbf{E}) \to \dots$$

in homotopy.

Proof. The sequence in question can be written as

$$\cdots \to \pi_{n+k}(\Omega E_{k+1}) \to \pi_{n+k}(F_k) \to \pi_{n+k}(G_k) \to \pi_{n+k}(E_{k+1}) \to \dots$$

for $k \geq n+2$. It is an exact sequence of abelian groups by Proposition 5.21. □

Proposition 7.79. *For a map $f : A \to X$ between pointed spaces the sequence* (25) *is a fibre sequence of spectra.*

Proof. We will not give the details. One first introduces a notion of a cofibre sequence of prespectra based on mapping cones. Then one observes that the mapping cone sequence (24) induces a cofibre sequence of prespectra. Finally one observes that the associated spectrum construction $R(-)$ turns cofibre sequences in spectra into fibre sequences. □

In order to simplify the notation we define

$$H^*(f;\mathbf{E}) := H^*(C(f);\mathbf{E})$$

and for an inclusion of a subspace $f : A \subseteq X$ we set

$$H^*(X,A;\mathbf{E}) := H^*(f;\mathbf{E}) \ .$$

Corollary 7.80. *For a map $f : A \to X$ we have a long exact homology sequence*

$$\cdots \to H_k(A;\mathbf{E}) \to H_k(X;\mathbf{E}) \to H_k(f;\mathbf{E}) \to H_{k-1}(A;\mathbf{E}) \to \dots$$

Assume that

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

is an increasing filtration of a space. We get an increasing filtration of the homology by

$$F_pH_*(X;\mathbf{E}) := \mathtt{im}(H_*(X_p;\mathbf{E}) \to H_*(X;\mathbf{E}))$$

for all $p \in \mathbb{Z}$. The associated graded groups can be calculated by a spectral sequence.

In order to obtain the usual grading conventions for spectral sequences it us useful to consider the decreasing filtration and a cohomological grading of homology

$$F^{-p}\mathcal{H}^{-q}(X;\mathbf{E}) := F_pH_q(X;\mathbf{E}) \ , \quad p,q \in \mathbb{Z} \ .$$

Then we can put the long exact pair sequences of the pairs (X_p, X_{p-1}) together and obtain an exact couple

$$\begin{tikzcd}
\bigoplus_j \mathcal{H}^*(X_j;\mathbf{E}) \arrow[rr] & & \bigoplus_j \mathcal{H}^*(X_j;\mathbf{E}) \arrow[dl] \\
& \bigoplus_j \mathcal{H}^*(X_{j+1},X_j;\mathbf{E}) \arrow[ul] &
\end{tikzcd} \ .$$

This exact couple induces a spectral sequence. The spectral sequence resides in the left half-plane. If it converges (e.g. if the filtration is finite), then we have

$$E_\infty^{p,q} \cong \mathrm{Gr}^p H_{-p-q}(X;\mathbf{E}) \ .$$

If X is a CW complex and the filtration is by its skeleta, the the spectral sequence is called the Atiyah-Hirzebruch spectral sequence. Its second page is

$$E_2^{p,q} \cong H_{-p}(X;E^q) \ .$$

If $\mathbf{E} = H\mathbb{Z}$, then the spectral sequence lives in the zero line. In this case the E_1-term

$$E_1^{-p,0} \cong C_p(X) \cong \bigoplus_{p-cells} \mathbb{Z}$$

is called the **cellular chain** complex $(C_*(X), \partial)$ of X, where $\partial = d_1$. Its differentials ∂ can be determined using the following Lemma:

Lemma 7.81. *The cellular cochain complex $C^*(X;A)$ of X (11) is obtained by applying $\mathtt{Hom}(-,A)$ to the cellular chain complex $(C_*(X),\partial)$.*

Proof. Excercise.

Example 7.82. We have

$$H_*(\mathbb{CP}^k;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 2,4,\dots,2k \\ 0 & else \end{cases}$$

Here we use that $\mathbb{CP}^n$ has cells in even dimensions $0,2,\dots,2n$. The complex is

$$\mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{Z} \,..$$

$$H_*(\mathbb{RP}^{2k-1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0, 2k-1 \\ \mathbb{Z}/2\mathbb{Z} & * = 1,3,\dots,2k-3 \\ 0 & else \end{cases}$$

Here the chain complex has the form

$$\mathbb{Z} \stackrel{0}{\leftarrow} \mathbb{Z} \stackrel{2}{\leftarrow} \mathbb{Z} \stackrel{0}{\leftarrow} \mathbb{Z} \stackrel{2}{\leftarrow} \mathbb{Z} \leftarrow \cdots \stackrel{0}{\leftarrow} \mathbb{Z} \,,$$

where the last $\mathbb{Z}$ is in degree $2k-1$.

The relation between complexes

$$C^*(X,A) \cong \mathtt{Hom}(C_*(X),\mathbb{Z})$$

is reflected in cohomology/homology by the **universal coefficient theorem UCT**. We start with the algebraic fact.

Lemma 7.83. *Let C_* be a chain complex consisting of free $\mathbb{Z}$-modules, A be some abelian group and define $C^* := \mathtt{Hom}(C_*,A)$. Then we have a short exact sequence*

$$0 \to \mathrm{Ext}(H_{k-1}(C_*),A) \to H^k(C^*) \to \mathtt{Hom}(H_k(C_*),A) \to 0 \;.$$

Proof. The chain complex C_* gives rise to a short exact sequence of chain complexes

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & Z_{k-1} \arrow[r] \arrow[u] & C_{k-1} \arrow[r, "\partial"] \arrow[u] & B_{k-2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z_k \arrow[r] \arrow[u, "0"] & C_k \arrow[r, "\partial"] \arrow[u, "\partial"] & B_{k-1} \arrow[r] \arrow[u, "0"] & 0 \\
0 \arrow[r] & Z_{k+1} \arrow[r] \arrow[u, "0"] & C_{k+1} \arrow[r, "\partial"] \arrow[u, "\partial"] & B_k \arrow[r] \arrow[u, "0"] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

Using that the Z_* are free we can take $\texttt{Hom}(-, A)$ and get a short exact sequence of cochain complexes

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 & Z^*_{k-1} \arrow[l] \arrow[d, "0"] & C^{k-1} \arrow[l] \arrow[d, "d"] & B^*_{k-2} \arrow[l, "d"'] \arrow[d, "0"] & 0 \arrow[l] \\
0 & Z^*_{k} \arrow[l] \arrow[d, "0"] & C^{k} \arrow[l] \arrow[d, "d"] & B^*_{k-1} \arrow[l, "d"'] \arrow[d, "0"] & 0 \arrow[l] \\
0 & Z^*_{k+1} \arrow[l] \arrow[d] & C^{k+1} \arrow[l] \arrow[d] & B^*_{k} \arrow[l, "d"'] \arrow[d] & 0 \arrow[l] \\
 & {} & {} & {} &
\end{tikzcd} ,$$

where $E^* = \texttt{Hom}(E, A)$. We now consider the long exact sequence in cohomology which gives

$$Z^*_{k-1} \to B^*_{k-1} \to H^k(C^*) \to Z^*_k \to B^*_k \ . \tag{28}$$

We have

$$0 \to B_k \to Z_k \to H_k(C_*) \to 0 \ .$$

Applying $\texttt{Hom}(-, A)$ and using that B_k is free we get

$$0 \to \texttt{Hom}(H_k(C_*), A) \to Z^*_k \to B^*_k \to \mathrm{Ext}(H_k(C_*), A) \to 0 \ .$$

This sequence describe the kernel and cokernel of the maps $Z^*_k \to B^*_k$ and $Z^*_{k-1} \to B^*_{k-1}$ in (28). We conclude that

$$0 \to \mathrm{Ext}(H_{k-1}(C_*), A) \to H^k(C^*) \to \texttt{Hom}(H_k(C_*), A) \to 0$$

is exact. $\square$

We get the **universal coefficient theorem**

Corollary 7.84. *If X is a pointed space, then for each $k \in \mathbb{N}$ we have a (functorial) short exact sequence*

$$0 \to \mathrm{Ext}(H_{k-1}(X; \mathbb{Z}), A) \to H^k(X; A) \to \texttt{Hom}(H_k(X; \mathbb{Z}), A) \to 0 \ .$$

Example 7.85. *We consider the case $\mathbb{RP}^3$. Since*

$$H_2(\mathbb{RP}^3; \mathbb{Z}) \cong 0 \ , \quad H_1(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

we have

$$H^2(\mathbb{RP}^3; \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

as expected.

7.7 Hurewicz

We will first construct a morphism of spectra

$$\epsilon : \mathbf{S} \to H\mathbb{Z} \ .$$

Here $H\mathbb{Z}$ consists of the spaces $H\mathbb{Z}_n = K(n, \mathbb{Z})$ for $n \geq 0$ and $*$ for $n < 0$. Note that $K(\mathbb{Z}, 0) \cong \mathbb{Z}$. For simplicity we use a version where $\Omega K(\mathbb{Z}, n+1) \cong K(\mathbb{Z}, n)$ (a homeomorphism and not just a homotopy equivalence).

Let $S^0 \to K(\mathbb{Z}, 0)$ be the map which sends to non-base point to 1. Using the structure map of the Eilenberg-MacLane spectrum $H\mathbb{Z}$ this map induces a map

$$\Sigma^n S^0 \to \Sigma^n K(\mathbb{Z}, 0) \to K(\mathbb{Z}, n)$$

for all $n \in \mathbb{N}$. For $0 \leq k \leq n$ we get an induced map

$$\Omega^{n-k} \Sigma^n S^0 \to \Omega^{n-k} K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, k) \ .$$

These maps are compatible in the sense that the squares

$$\begin{tikzcd}
\Omega^{n-k}\Sigma^n S^0 \arrow[r] \arrow[d] & K(\mathbb{Z}, k) \arrow[d, "\cong"] \\
\Omega^{n+1-k}\Sigma^{n+1} S^0 \arrow[r] & \Omega K(\mathbb{Z}, k+1)
\end{tikzcd}$$

commute. We get a collection of maps

$$S_k := \mathtt{colim}_n \Omega^{n-k} \Sigma^n S^0 \to K(\mathbb{Z}, k) \ , \quad k \in \mathbb{N}$$

where S_k are the constituents of the sphere spectrum $\mathbf{S}$. The collection of these maps is compatible with the structure maps and therefore provide a map between spectra

$$\epsilon : \mathbf{S} \to H\mathbb{Z} \ .$$

In induces a natural transformation between the homology theories

$$\epsilon : H_*(-; \mathbf{S}) \to H_*(-; \mathbb{Z}) \ .$$

The stable homotopy homology theory $H_*(-; \mathbf{S})$ gives quite strong information about spaces but is very difficult to compute since even its coefficients are not known. On the other hand, the ordinary integral homology $H_*(-; \mathbb{Z})$ is easy to compute using e.g. the cellular chain complex but loses more information. The strongest information about spaces are the (unstabilized) homotopy groups. They detect homotopy equivalences between CW-complexes. But they even more difficult to compute since they are not graded components of a homology theory.

The Hurewicz homomorphism compares homotopy and homology. We have a stabilization map

$$s : \pi_k(X) \to \mathtt{colim}_n \pi_{k+n}(\Sigma^n X) \cong H_k(X;\mathbf{S}) =: \pi^s_k(X) \ .$$

It is an a sense the best approximation of homotopy by a homology theory. The composition

$$h : \pi_k(X) \stackrel{s}{\to} H_k(X;\mathbf{S}) \stackrel{\epsilon}{\to} H_k(X;\mathbb{Z})$$

is called the **Hurewicz homomorphism.** It is natural in X.

Theorem 7.86. *If X is $n-1$-connected, then*

$$H_k(X;\mathbb{Z}) \cong 0 \ , \quad k < n$$

and

$$h : \pi_n(X) \to H_n(X;\mathbb{Z})$$

is an isomorphism (abelianization for $k=1$).

Proof. We have

$$H_k(X;\mathbb{Z}) \cong \mathtt{colim}_m \pi_m(X \wedge K(m-k;\mathbb{Z})) \ .$$

Since $K(m-k;\mathbb{Z})$ is $m-k-1$-connected and X is $n-1$-connected, by Proposition 6.22 the product $X \wedge K(m-k;\mathbb{Z})$ is $n+m-k$-connected for sufficiently large m. Hence $\pi_m(X \wedge K(m-k;\mathbb{Z})) = 0$ for $k \le n-1$ and large m.

Assume that X is a CW-complex. We get a morphism of Atiyah-Hirzebruch spectral sequences

$${}^{\mathbf{S}}E_r \to H\mathbb{Z}^{\mathbf{S}}E_r \ .$$

Since $\mathbf{S}^0 = H\mathbb{Z}^0 \cong \mathbb{Z}$ and both spectra have no negative homotopy the only term which contributes to $H_n(X;-)$ is $E_2^{-n,0}$. No differential can end there since the spectra sequence lives in the lower left quadrant. No differential can start there since the lower homology of X (even with coefficients in $\pi^s_*(\mathbf{S})$) vanishes. Since $\pi^s_0(\mathbf{S}) \cong \mathbb{Z}$ we have

$${}^{\mathbf{S}}E_2^{-n,0} \cong H_n(X;\mathbb{Z}) \ .$$

From the comparison of spectral sequences we thus get an isomorphism

$$\epsilon : H_n(X;\mathbf{S}) \cong H_n(X;H\mathbb{Z}) \ .$$

It remains to study the stabilization map

$$\pi_n(X) \to \pi^s_n(X) = \mathtt{colim}_k \pi_{n+k}(\Sigma^k X) \ .$$

If $n \ge 1$, the Freudenthal's theorem Theorem 6.24 applies and implies that this map is an isomorphism, too.

It remains to study the case $n = 1$ and the map

$$\pi_1(X) \to \pi_2(\Sigma X)$$

for a connected space. By Theorem 6.24 this map is surjective and since the target is abelian we get a surjective map

$$\pi_1(X) \to \pi_1(X)^{ab} \to \pi_2(\Sigma X) \ .$$

We refer to [tD08, Thm. 9.2.1] for an argument that $\pi_1(X)^{ab} \to \pi_2(X)$ is an isomorphism.

8 Aufgaben

1. Show that a covering has the unique homotopy lifting property for every space.

2. Show that the concatenation of homotopy classes of paths is an associative operation.

3. Let $\Phi : C \to D$ be an equivalence of categories. Show that the functor $\Phi^* : \mathbf{Fun}(D, \mathbf{Set}) \to \mathbf{Fun}(C, \mathbf{Set})$ given by precomposition is an equivalence of categories.

4. Verify in detail, that the functor

$$\Phi : \mathbf{Cov}(Y) \to \mathbf{Fun}(\Pi(Y), \mathbf{Set})$$

is well-defined.

5. Let $a : H \to G$ be an equivalence of groupoids and $C : G \to \mathbf{Set}$ be a representation. Show that the natural morphism $\tilde{a} : T(a^*C) \to T(C)$ between transport categories is an equivalence.

6. Let π and G be groups. Show that a π-Set S with a right G-action $G \to \mathrm{Aut}_{\pi\mathbf{Set}}(S)$ such that S is a right-G-torsor is isomorphic to one obtained from a homomorphism $\pi \to G$.

7. Let $S^1 \vee S^1$ be the wegde of two circles (obtained from the disjoint union $S^1 \sqcup S^1$ by identifying distinguished points in each copy. Let $a, b \in \pi_1(S^1 \vee S^1, *)$ be the elements represented by the two circles. Show that $\pi_1(S^1 \vee S^1, *)$ is freely generated by a, b.

Hint: Describe first explicitly a universal covering.

8. Let (X, x) and (Y, y) be pointed space. Then we form the pointed space $(X \times Y, (x, y))$. Show that the natural map

$$\pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y)$$

induced by the projections to the factors induces an isomorphism of groups.

9. Show that $\mathbb{CP}^n$ is simply connected for every $n \geq 1$.

10. Let $S^1 \to S^3$ be a smooth embedding. Show that $S^3 \setminus S^1$ is not simply connected.

11. Let ω be a closed one-form on a connected manifold M and $m \in M$. On the set of smooth paths starting in m we define the relation

$$\gamma \sim \mu \quad := \quad \int_\gamma \omega = \int_\mu \omega \;\; and \;\; \gamma(1) = \mu(1) \ .$$

Show that this is an equivalence relation. Let X denote the set of equivalence classes of such paths and $f : X \to M$ be the the map $[\gamma] \mapsto \gamma(1)$. Show that X has a natural manifold structure such that f is a covering and $f^*\omega$ is exact.

12. We continue the previous exercise. Let $g : Z \to M$ be a smooth map such that $g^*\omega$ is exact. Show that there exists a smooth lift h in

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[ur, dotted, "h"] \arrow[r, "g"] & M \end{tikzcd} \ .$$

13. A flat vector bundle is called irreducible if it does not contain a non-trivial proper flat subbundle. Describe the set of isomorphism classes of flat two-dimensional real vector bundles on T^2 explicitly.

14. Let G be a topological group. On $\pi_1(G, 1)$ we consider the two group structures:

1. $[\gamma] \circ [\mu] := [\gamma \circ \mu]$ given by concatenation of paths,
2. $[\gamma] * [\mu] := [\gamma\mu]$ given by pointwise multiplication of paths.

Show that $* = \circ$ and that $\pi_1(G, 1)$ is abelian.

15. Let X and Y be locally compact Hausdorff spaces and Z be any topological space. Show that there is a natural homeomorphism

$$\mathtt{Map}(X \times Y, Z) \cong \mathtt{Map}(X, \mathtt{Map}(Y, Z)) \ .$$

Further the composition

$$\mathtt{Map}(X, Y) \times \mathtt{Map}(Y, Z) \to \mathtt{Map}(X, Z)$$

is continuous.

16. Let G be a group which acts on a topological space X. Then we can form the quotient X/G by the usual universal property. If Y is a locally compact Hausdorff space, then we let G act on $X \times Y$ via its action on the first factor. Show that have a homeomorphism

$$(X \times Y)/G \cong (X/G) \times Y \ .$$

17. Let G be a topological group and LG be its loop group. Their base points are the identity elements, respectively. Show that for every $n \in \mathbb{N}$ we have an isomorphism of groups $\pi_n(LG) \cong \pi_n(G) \times \pi_{n+1}(G)$.

18. Let X be a pointed space and $[\ldots, X] : h\mathbf{Top}_*^{op} \to \mathbf{Set}_*$ be the pointed set valued functor represented by X. Show that a refinement of this functor to a monoid valued functor determines an H-space structure on X which is unique up to isomorphism.

19. We let $\mathbb{Z}_p^\delta$ and $\mathbb{Z}_p$ denote the sets of p-adic numbers with the discrete, and with usual locally compact topology, respectively. Show that the map $\mathtt{id}_{\mathbb{Z}_p} : \mathbb{Z}_p^\delta \to \mathbb{Z}_p$ is a weak equivalence, but not a homotopy equivalence.

20. Let $f : X \to Y$ be a morphism of co-H-spaces. Show that $C(f)$ has a naturally induced co-H-space structure.

21. A map $i : A \to X$ is called a **cofibration** if it has the homotopy extension property for all spaces Y, i.e. if for every diagram (the bold part given)

$$\begin{tikzcd} I \times A \cup \{0\} \times X \arrow[r, "H"] \arrow[d] & Y \\ I \times X \arrow[ur, dotted, "\tilde{H}"] & \end{tikzcd}$$

the extension $\tilde{H}$ exists. Show that if i is a neighbourhood deformation retract, then it is a cofibration.

22. Recall that the mapping cylinder $Z(f)$ of a map $f : X \to Y$ is defined by $I \times X \cup_f Y$, where we identify $[1, x] \in I \times X$ with $f(x) \in Y$. We consider the inclusion $i : X \cong \{0\} \times X \to Z(f)$.

1. Show that $f : X \to Y$ is homotopy equivalent to $i : X \to Z(f)$ in $\mathbf{Top}^{\Delta^1}$.
2. Show that $i : X \to Z(f)$ is a neighbourhood deformation retract.
3. Show that the inclusion $i : X \to Z(f)$ is a cofibration (see Aufgabe 1).

23. Show that if $i : A \to X$ is a cofibration (see Aufgabe 1), then the canonical map $C(i) \to X/A$ is a homotopy equivalence.

24. We consider pointed spaces X and Y.

1. Show that for a map $f : X \to Y$ the map $\Sigma f : X \to Y$ is a map of co-H-spaces.
2. Show that the involution $- : \Sigma X \to \Sigma X$ represents the co-inverse.
3. Show that a map $\Sigma X \to \Sigma Y$ may not be a map of co-H-spaces.

25. Let $f : X \to Y$ be a fibration and y_0 and y_1 two points. Show that $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are homotopy equivalent provided that y_0 and y_1 belong to the same connected component.

26. Show that

$$f : \mathbb{R} \to \mathbb{R} , \quad f(x) := x^2$$

is not a fibration.

27. We consider a pull-back square in **Top**

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd} .$$

Show that if f is a fibration, then f' is a fibration.

28. Let $f : X \to Y$ be a fibration and T be locally compact. Show that then the induced map $\mathtt{Map}(T, X) \to \mathtt{Map}(T, Y)$ is a fibration.

29. Read Theorem 7.1 in Husemoller's book [Hus94] on fibre bundles. Conclude that a locally trivial fibre bundle is a quasi-fibration.

30. Let $f, g : S^n \to S^m$ be continuous maps. Show that

$$\deg(f) \deg(g) = \deg(f \wedge g) .$$

Further show in the case that $n = m$ that

$$\deg(f \circ g) = \deg(f) \deg(g) .$$

31. Calculate the homotoyp groups $\pi_*(\mathbb{RP}^n)$. Conclude that

$$\mathbb{RP}^\infty := \mathrm{colim}_n \mathbb{RP}^n \simeq K(\mathbb{Z}/2\mathbb{Z}, 1)\ .$$

32. Let $f : D^n \to \mathbb{R}^n$ be such that $f_{|S^{n-1}} = \mathtt{id}_{S^{n-1}}$. Show that then $D^n \subseteq \mathtt{im}(f)$.

33. Let I be a small category and $\mathcal{X} \in \mathbf{Top}_*^I$ be an I-diagram of topological spaces. Analyse under which conditions

$$\pi_n(\mathtt{colim}_I \mathcal{X}) \cong \mathtt{colim}_I \pi_n(\mathcal{X})$$

and

$$\pi_n(\mathtt{lim}_I \mathcal{X}) \cong \mathtt{lim}_I \pi_n(\mathcal{X})\ .$$

34. Construct a non-trivial action of $\pi_1(X)$ on $\pi_n(X)$ which is natural on X.

35. Let X be a CW-complex and $X_n \subseteq X$ be its n-skeleton. Show that the inclusion $X_n \to X$ is n-connected.

36. Show that $\pi_3(SO(n))$ is infinite for $n \geq 3$.

37. For $n \in \mathbb{N}$ consider the simplicial set $\partial\Delta^n \in \mathbf{sSet}$ given as a union of the images of $\Delta(\partial_i) : \Delta^{n-1} \to \Delta^n$ for all i. Show that $|\partial\Delta^n| \cong S^{n-1}$.

38. Determine the non-degenerated simplices of $\Delta^1 \times \Delta^1$. Verify explicitly, that $|\Delta^1 \times \Delta^1| \cong |\Delta^1| \times |\Delta^1|$.

39. Describe a simplicial set X such that $|X| \cong T^2$.

40. Let G be a topological group. Show that $|\mathtt{sing}^{top}(G)|$ is again a topological group which comes with a canonical homomorphism to G.

References

[Hus94] Dale Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994.

[Spa81] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[tD08] Tammo tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.